Polynomials	Misha Lavrov

How to Solve Polynomials: Solutions

Western PA ARML Practice — *January 17, 2016*

1. *(ARML 1977) Find the remainder that results when* $(x+1)^5+(x+2)^4+(x+3)^3+(x+4)^2+(x+5)$ *is divided by* $x+2$.

   The remainder can be obtained by substituting $x=-2$, getting $(-1)^5+0^4+1^3+2^2+3=7$.

2. *(ARML 1978) Find the smallest root of* $(x-3)^3+(x+4)^3=(2x+1)^3$.

   Noting that $(x-3)+(x+4)=(2x+1)$, the equation has the form $A^3+B^3=(A+B)^3$. Simplifying $(A+B)^3-A^3-B^3=0$ gives $AB(A+B)=0$, so either A, B, or $A+B$ is 0.

   Therefore the roots of the equation are the roots of $x-3=0$, $x+4=0$, and $2x+1=0$: 3, -4, and $\frac{1}{2}$. Of these, -4 is the smallest.

3. *(ARML 2010) Compute all ordered pairs of real numbers* (x,y) *that satisfy both of the equations:*

   $$x^2+y^2=6y-4x+12 \qquad \text{and} \qquad 4y=x^2+4x+12.$$

   The second equation gives us $x^2=4y-4x-12$, which (substituted into the first) gets us to $4y-4x-12+y^2=6y-4x+12$, or $y^2+4y-12=6y+12$, or $y^2-2y-24=0$. This factors as $(y-6)(y+4)=0$, so $y=-4$ or $y=6$.

   When $y=-4$, we have $x^2+4x+12=-16$ from the second equation; $x^2+4x+28=0$ has no real solutions.

   When $y=6$, the same equation becomes $x^2+4x+12=24$, or $x^2+4x--12=0$. This factors as $(x-2)(x+6)=0$, so we get the two solutions $(x,y)=(2,6)$ and $(x,y)=(-6,6)$.

4. *(ARML 1980) Find the real value of* x *which satisfies*

   $$x^3+(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3+(x-5)^3=3^3.$$

   It's possible that you will just guess $x=3$ and spot that all terms on the left-hand side cancel except $x^3=3^3$.

   If not, $x=3$ can also be found with the rational root theorem. The constant term, when all is simplified, is 252, which has lots of factors; the leading coefficient is 6, which doesn't help.

   (What might help is the depressing substitution $y=x+\frac{5}{2}$, which produces a polynomial with smaller coefficients; but this doesn't help *much*.)

5. *(ARML 1979) Two of the solutions of*

   $$x^4-3x^3+5x^2-27x-36=0$$

   are pure imaginary numbers. Find these two solutions.

Substituting $x = yi$ into the equation gives us

$$y^4 + 3iy^3 - 5y^2 - 27yi - 36 = 0 \qquad Leftrightarrow \qquad (y^4 - 5y^2 - 36) + (3y^3 - 27y)i = 0.$$

So $y^4 - 5y^2 - 36 = 0$ and $3y^3 - 27y = 0$. The first equation factors as $(y^2+4)(y^2-9) = 0$, and the second equation factors as $y(y^2 - 9) = 0$, which have the common roots $y = \pm 3$.

Therefore the two pure imaginary solutions are $x = \pm 3i$.

6. *(ARML 1978) Find the four values of x which satisfy $(x-3)^4 + (x-5)^4 = -8$.*

   The depressing substitution $x = y + 4$ gives

   $$(y-1)^4 + (y+1)^4 = -8 \qquad \Leftrightarrow \qquad 2y^4 + 12y^2 + 10 = 0,$$

   which factors as $2(y^2+1)(y^2+5) = 0$. This has roots $y = \pm i, \pm i\sqrt{5}$, so we get the solutions $x = 4+i, 4-i, 4+i\sqrt{5}, 4-i\sqrt{5}$ to the original equation.

7. *(ARML 1994) If $x^5 + 5x^4 + 10x^3 + 10x^2 - 5x + 1 = 10$, and $x \neq -1$, compute the numerical value of $(x+1)^4$.*

   The left-hand side should be recognized as $(x+1)^5 - 10x$, since almost all the coefficients are positive binomial coefficients. Then we have $(x+1)^5 = 10x + 10 = 10(x+1)$. Having $x \neq -1$ allows us to divide both sides by $x+1$, obtaining $(x+1)^4 = 10$.

8. *(ARML 1991) If $(x^2+x+1)(x^6+x^3+1) = \frac{10}{x-1}$, compute the real value of x.*

   Multiplying both sides by $x - 1$ lets us simplify $(x^2+x+1)(x-1)$ to $x^3 - 1$ and then $(x^6 + x^3 + 1)(x^3 - 1)$ to $x^9 - 1$, getting $x^9 - 1 = 10$, or $x^9 = 11$. This has only one real root: $x = \sqrt[9]{11}$.

9. We have

   $$\begin{aligned} x^n - y^n &= (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}) \\ x^n + y^n &= (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - xy^{n-2} + y^{n-1}) \end{aligned}$$

   (the second only when n is odd).

10. *(ARML 1987) The equation $x^4 - 3x^3 - 6 = 0$ has exactly two real roots, r and s. Compute $\lfloor r \rfloor + \lfloor s \rfloor$.*

    Since we only need to know the floor of the roots, it suffices to figure evaluate the function $f(x) = x^4 - 3x^3 - 6 = 0$ at a few integer coordinates:

x	-2	-1	0	1	2	3	4
$f(x)$	34	-2	-6	-8	-14	-6	58

    There is a sign change from $x = -2$ to $x = -1$, and another sign change from $x = 3$ to $x = 4$. Therefore both intervals contain a root, and $\lfloor r \rfloor + \lfloor s \rfloor = -2 + 3 = 1$.

11. *(ARML 1992) Compute the positive integer value of k that makes the following statement true:*

For all positive integers a, b, and c that make the roots of $ax^2+bx+c=0$ rational, the roots of $4ax^2+12bx+kc=0$ will also be rational.

If $ax^2+bx+c=0$ has rational roots, so does $a(x/3)^2+b(x/3)+c=0$, or $\frac{1}{9}ax^2+\frac{1}{3}bx+c=0$. This has the same roots as $36(\frac{1}{9}ax^2+\frac{1}{3}bx+c)=0$, or $4ax^2+12bx+36c=0$.

Therefore the statement is true for $k=36$.

12. *(ARML 2000) Let $f(x)=(x-1)(x-2)^2(x-3)^3\cdots(x-1999)^{1999}(x-2000)^{2000}$. Compute the number of values of x for which $|f(x)|=1$.*

    What, you thought I would tell you everything?