Polynomials — Misha Lavrov

How to Solve Polynomials

Western PA ARML Practice — *January 17, 2016*

Warm-up

(ARML 1990) Compute $\dfrac{1990^3 - 1000^3 - 990^3}{1990 \cdot 1000 \cdot 990}$.

Facts to know

Remainder theorem. If $f(x)$ is a polynomial with $f(a) = r$, then $f(x)$ can be written as $(x-a)g(x) + r$ for some polynomial g.

Rational root theorem. All the rational solutions to the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

(with integer coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$) have the form $\pm\frac{p}{q}$, where p is a divisor of a_0 and q is a divisor of a_n.

Depressed polynomial. Often it is easier to solve a polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

by first making the substitution $x = y - \frac{a_{n-1}}{n \cdot a_n}$. When you expand, this should result in an equation with no y^{n-1} term.

(This is not always useful, but it occasionally manages to sidestep a number of other "clever tricks". It works best when $\frac{a_{n-1}}{n \cdot a_n}$ is an integer, for obvious reasons.)

Useful identities. The following factorizations and expansions are good to know, at a minimum:

$$\begin{aligned}
x^2 - y^2 &= (x+y)(x-y) \\
x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3.
\end{aligned}$$

Horner's method. It is convenient to evaluate a polynomial by writing it in the form

$$f(x) = a_0 + x(a_1 + x(a_2 + x(a_3 + \cdots + x(a_{n-1} + xa_n)))).$$

To evaluate, set $b_n = a_n$, $b_{n-1} = a_{n-1} + xb_n$, and repeat by setting $b_i = a_i + xb_{i+1}$. Then b_0 will be equal to $f(x)$.

(Here, b_i evaluates the part of $f(x)$ contained inside i layers of parentheses.)

Problems

1. (ARML 1977) Find the remainder that results when $(x+1)^5+(x+2)^4+(x+3)^3+(x+4)^2+(x+5)$ is divided by $x+2$.
2. (ARML 1978) Find the smallest root of $(x-3)^3+(x+4)^3=(2x+1)^3$.
3. (ARML 2010) Compute all ordered pairs of real numbers (x, y) that satisfy both of the equations:
$$x^2+y^2=6y-4x+12 \qquad \text{and} \qquad 4y=x^2+4x+12.$$
4. (ARML 1980) Find the real value of x which satisfies
$$x^3+(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3+(x-5)^3=3^3.$$
5. (ARML 1979) Two of the solutions of
$$x^4-3x^3+5x^2-27x-36=0$$
are pure imaginary numbers. Find these two solutions.
6. (ARML 1978) Find the four values of x which satisfy $(x-3)^4+(x-5)^4=-8$.
7. (ARML 1994) If $x^5+5x^4+10x^3+10x^2-5x+1=10$, and $x \neq -1$, compute the numerical value of $(x+1)^4$.
8. (ARML 1991) If $(x^2+x+1)(x^6+x^3+1)=\frac{10}{x-1}$, compute the real value of x.
9. The factorizations of $x^3 \pm y^3$ on the other side of the page have their generalizations.
   (a) Similarly to the factorization of x^3-y^3, you can factor $x-y$ out of x^n-y^n for any n. (Why?) What is the resulting identity?
   (b) Similarly to the factorization of x^3+y^3, you can factor $x+y$ out of x^n+y^n for any *odd* n. (Why?) What is the resulting identity?
10. (ARML 1987) The equation $x^4-3x^3-6=0$ has exactly two real roots, r and s. Compute $\lfloor r \rfloor + \lfloor s \rfloor$.
11. (ARML 1992) Compute the positive integer value of k that makes the following statement true:

    For all positive integers a, b, and c that make the roots of $ax^2+bx+c=0$ rational, the roots of $4ax^2+12bx+kc=0$ will also be rational.
12. (ARML 2000) Let $f(x)=(x-1)(x-2)^2(x-3)^3\cdots(x-1999)^{1999}(x-2000)^{2000}$. Compute the number of values of x for which $|f(x)|=1$.