Polynomials — Misha Lavrov

How to Not Solve Polynomials: Solutions

Western PA ARML Practice — *January 24, 2016*

1. **(ARML 1996) The roots of $ax^2 + bx + c = 0$ are irrational, but their calculator approximations are 0.8430703308 and −0.5930703308. Compute the integers a, b, and c.**

   We compute $0.843 + -0.593 \approx \frac{1}{4}$ and $0.843 \cdot -0.593 \approx -\frac{1}{2}$, so $x^2 - \frac{1}{4}x - \frac{1}{2} = 0$ would work, or $4x^2 - x - 2 = 0$ if we want the coefficients to be integers.

2. **Let a and b be the roots of $x^2 - 3x - 1 = 0$.**

   (a) **Find a quadratic equation whose roots are a^2 and b^2.**

   We have $a^2 + b^2 = (a+b)^2 - 2ab = 3^2 + 2 = 11$, and $a^2 \cdot b^2 = (ab)^2 = 1$, so the quadratic equation $x^2 - 11x + 1 = 0$ works.

   (b) **Find a quadratic equation whose roots are $\frac{1}{a}$ and $\frac{1}{b}$.**

   We could compute $\frac{1}{a} \cdot \frac{1}{b}$ and $\frac{1}{a} + \frac{1}{b}$, but easier is to substitute $\frac{1}{x}$ for x and rewrite $\frac{1}{x^2} - \frac{3}{x} - 1 = 0$ as $1 - 3x - x^2 = 0$, or $x^2 + 3x - 1 = 0$.

   (c) **Find a quadratic equation whose roots are $\frac{1}{a+1}$ and $\frac{1}{b+1}$, and use this to compute $\frac{1}{a+1} + \frac{1}{b+1}$.**

   Similarly, the equation $(\frac{1}{x} - 1)^2 - 3(\frac{1}{x} - 1) - 1 = 0$ will have the right roots; this expands to $3x^2 - 5x + 1 = 0$. Here, the sum of the roots is $\frac{5}{3}$.

3. **(HMMT 1998) Three of the roots of $x^4 + ax^2 + bx + c = 0$ are −2, −3, and 5. Find the value of $a + b + c$.**

   Since the equation has no x^3 term, the sum of the roots is 0, so the fourth root must be 0. The polynomial factors as $x(x+2)(x+3)(x-5)$, which yields −48 when evaluated at $x = 1$. But evaluating the polynomial at $x = 1$ can also be done in its original form, getting $1 + a + b + c$. Therefore $a + b + c = -49$.

4. **(AIME 2005) The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Find their sum (as a fraction $\frac{m}{n}$).**

   Let $y = 2^{111x}$; then we have $\frac{1}{4}y^3 + 4y = 2y^2 + 1$. This is a cubic equation with three roots, whose product is 4. Therefore if the original equation has roots r_1, r_2, r_3, we have $2^{111(r_1+r_2+r_3)} = 4 = 2^2$, which means $r_1 + r_2 + r_3 = \frac{2}{111}$.

5. **(ARML 1983) Let a, b, and c be the sides of triangle ABC. If a^2, b^2, and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$ (where P, Q, and R are constants), express**

   $$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

   in terms of one or more of the coefficients P, Q, and R.

By the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$, so $\cos C = \frac{a^2+b^2-c^2}{2ab}$. Writing similar expressions for $\cos A$ and $\cos B$, we get

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc}.$$

From the coefficients P, Q, and R, we get $a^2+b^2+c^2 = P$ and $a^2b^2c^2 = R$, so $\frac{a^2+b^2+c^2}{2abc} = \frac{P}{2\sqrt{R}}$.

6. **(AIME 2001) Find the sum of all the roots of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$.**

   Though Vieta's formulas can be used here, it's easier to observe that if $x = r$ is a solution, so is $x = \frac{1}{2} - r$. Therefore, the 2000 roots of this degree-2000 equation come in 1000 pairs which add up to $\frac{1}{2}$, and so their sum is 500.

7. **(ARML 2006) The two equations $y = x^4 - 5x^2 - x + 4$ and $y = x^2 - 3x$ intersect at four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Compute $y_1 + y_2 + y_3 + y_4$.**

   Since all four points lie on the second curve, we can write $y_1 + y_2 + y_3 + y_4$ as $x_1^2 + x_2^2 + x_3^2 + x_4^2 - 3(x_1 + x_2 + x_3 + x_4)$.

   Moreover, x_1, x_2, x_3, x_4 are solutions to $x^4 - 5x^2 - x + 4 = x^2 - 3x$, or $x^4 - 6x^2 + 2x + 4 = 0$. So we have $x_1+x_2+x_3+x_4 = 0$, and $x_1^2+x_2^2+x_3^2+x_4^2 = (x_1+x_2+x_3+x_4)^2 - 2(x_1x_2 + \cdots + x_3x_4) = 0 - 2 \cdot -6 = 12$. Thus, $y_1 + y_2 + y_3 + y_4 = 12$.

8. **(ARML 2010) For real numbers α, B, and C, the roots of $T(x) = x^3 + x^2 + Bx + C$ are $\sin^2\alpha$, $\cos^2\alpha$, and $-\csc^2\alpha$. Compute $T(5)$.**

   The coefficient of x^2 in $T(x)$ is the negative sum of the roots, so we have $\sin^2\alpha + \cos^2\alpha - \csc^2\alpha = -1$. Since $\sin^2\alpha + \cos^2\alpha = 1$ for all α, this means $\csc^2\alpha = 2$, so $\sin^2\alpha = \cos^2\alpha = \frac{1}{2}$. Therefore $T(x) = (x - \frac{1}{2})^2(x+2)$, and $T(5) = (5 - \frac{1}{2})^2(5+2) = \frac{567}{4}$.

9. **The roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c. The roots of $x^3 + rx^2 + sx + t = 0$ are $a+b$, $b+c$, and $c+a$.**

   (a) **Find r.**

   We have $a + b + c = -3$, and $r = -(a+b) - (a+c) - (b+c) = -2(a+b+c) = 6$.

   (b) **(AIME 1996) Find t.**

   Since $a+b+c = -3$, we have $a+b = -c-3$, $b+c = -a-3$, and $c+a = -b-3$. Since $t = -(a+b)(b+c)(c+a)$, we have $t = (a+3)(b+3)(c+3)$, which is the negative of the original polynomial evaluated at $x = -3$. We conclude $t = 23$.

10. **(AIME 2004) Let C be the coefficient of x^2 in the product**

$$(1-x)(1+2x)(1-3x)(\cdots)(1+14x)(1-15x).$$

   Find $|C|$.

   Equivalently, consider the coefficient of x^{13} in the polynomial $(x-1)(x+2)(x-3)(\cdots)(x+14)(x-15)$. This polynomial has roots $1, -2, 3, -4, \ldots, -14, 15$, whose sum is 8 and whose sum of squares is 1240 by the well-known formula. So the sum of products of the roots taken two at a time is $\frac{8^2-1240}{2} = -588$, giving $|C| = 588$.