Polynomials	Misha Lavrov
How to Not Solve Polynomials	
Western PA ARML Practice	*January 24, 2016*

Warm-up

A rectangle's perimeter is 16 and the length of a diagonal is 6. What is the area of the rectangle?

Facts to know

Vieta's Formula (Simple). If the quadratic equation $x^2 + px + q = 0$ has roots r_1 and r_2, then

$$r_1 + r_2 = -p \qquad \text{and} \qquad r_1 \cdot r_2 = q.$$

Vieta's Formula (Cubic). Just to give you another special case to warm up before the general case: if the cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots r_1, r_2, r_3, then

$$\begin{cases} -\frac{b}{a} = r_1 + r_2 + r_3, \\ +\frac{c}{a} = r_1 r_2 + r_1 r_3 + r_2 r_3, \\ -\frac{d}{a} = r_1 r_2 r_3. \end{cases}$$

Vieta's Formula (General). If the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ has roots $r_1, r_2, \cdots, r_n$ (listed with multiplicity), then:

$$\begin{cases} -\frac{a_{n-1}}{a_n} & = r_1 + r_2 + \cdots + r_n, \\ +\frac{a_{n-2}}{a_n} & = r_1 r_2 + r_1 r_3 + \cdots + \cdots + r_{n-1} r_n, \\ -\frac{a_{n-3}}{a_n} & = r_1 r_2 r_3 + r_1 r_2 r_4 + \cdots + \cdots + \cdots + r_{n-2} r_{n-1} r_n, \\ & \vdots \\ (-1)^{n-1} \frac{a_1}{a_n} & = r_1 r_2 \cdots r_{n-1} + r_1 r_2 \cdots r_{n-2} r_n + \cdots + r_2 r_3 \cdots r_n, \\ (-1)^n \frac{a_0}{a_n} & = r_1 r_2 \cdots r_n. \end{cases}$$

In general, $(-1)^k \frac{a_{n-k}}{a_n}$ will be the sum of all possible k-fold products of the roots.

The first and the last equations are the most useful, giving the sum and the product of the roots, respectively.

Newton's identity. In the same notation as above, $r_1^2 + r_2^2 + \cdots + r_n^2 = \frac{a_{n-1}^2}{a_n^2} - 2\frac{a_{n-2}}{a_n}$.

Symmetry principle. Generally, we can compute a function $f(r_1, r_2, \ldots, r_n)$ without solving the polynomial equation if the function is symmetric in $r_1, r_2, \ldots, r_n$ (switching any two roots leaves it unchanged) and also not too terrible.

Problems

1. (ARML 1996) The roots of $ax^2+bx+c=0$ are irrational, but their calculator approximations are 0.8430703308 and -0.5930703308. Compute the integers a, b, and c.

   (You may assume that a is positive, that a, b, c are small (at most 10, say, in absolute value), and that they are not all three divisible by a common factor.)

2. Let a and b be the roots of $x^2 - 3x - 1 = 0$. *(It's more fun to solve this problem without computing a and b.)*

   (a) Find a quadratic equation whose roots are a^2 and b^2.

   (b) Find a quadratic equation whose roots are $\frac{1}{a}$ and $\frac{1}{b}$.

   (c) Find a quadratic equation whose roots are $\frac{1}{a+1}$ and $\frac{1}{b+1}$, and use this to compute $\frac{1}{a+1} + \frac{1}{b+1}$.

3. (HMMT 1998) Three of the roots of $x^4 + ax^2 + bx + c = 0$ are -2, -3, and 5. Find the value of $a + b + c$.

4. (AIME 2005) The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Find their sum (as a fraction $\frac{m}{n}$).

5. (ARML 1983) Let a, b, and c be the sides of triangle ABC. (An assumption which ARML sometimes takes without saying is that a is the length of BC, which is opposite $\angle A$, b is opposite $\angle B$, and so on.)

   If a^2, b^2, and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$ (where P, Q, and R are constants), express
   $$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$
   in terms of one or more of the coefficients P, Q, and R.

6. (AIME 2001) Find the sum of all the roots of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$.

7. (ARML 2006) The two equations $y = x^4 - 5x^2 - x + 4$ and $y = x^2 - 3x$ intersect at four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Compute $y_1 + y_2 + y_3 + y_4$.

8. (ARML 2010) For real numbers α, B, and C, the roots of $T(x) = x^3 + x^2 + Bx + C$ are $\sin^2\alpha$, $\cos^2\alpha$, and $-\csc^2\alpha$. Compute $T(5)$.

9. The roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c. The roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$.

   (a) Find r.

   (b) (AIME 1996) Find t.

10. (AIME 2004) Let C be the coefficient of x^2 in the product
    $$(1 - x)(1 + 2x)(1 - 3x)(\cdots)(1 + 14x)(1 - 15x).$$
    Find $|C|$.