Solving recurrences

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ARML Practice 2/2/2014

Warm-up / Review

1. Compute

$$\prod_{k=2}^{100}\left(1-\frac{1}{k}\right)=\left(1-\frac{1}{2}\right)\times\left(1-\frac{1}{3}\right)\times\cdots\times\left(1-\frac{1}{100}\right).$$

2. Compute

$$\prod_{k=2}^{100}\left(1-\frac{1}{k^2}\right).$$

Homework: find and solve problem Algebra #7 from the February 2009 HMMT.

Warm-up / Review

Solutions

1. Write $1 - \frac{1}{k}$ as $\frac{k-1}{k}$. Then the product becomes

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \dots \cdot \frac{98}{99} \cdot \frac{99}{100}.$$

Except for the 1 and the 100, every number occurs once in the numerator and once in the denominator, so the final answer is $\frac{1}{100}$.

2. Write $1 - \frac{1}{k^2}$ as $\frac{k^2-1}{k^2} = \frac{k-1}{k} \cdot \frac{k+1}{k}$. Then the product becomes

$$\left(\frac{1}{2} \cdot \frac{3}{2}\right) \cdot \left(\frac{2}{3} \cdot \frac{4}{3}\right) \cdot \left(\frac{3}{4} \cdot \frac{5}{4}\right) \cdot \dots \cdot \left(\frac{99}{100} \cdot \frac{101}{100}\right).$$

The second factor of each term is the reciprocal of the first factor of the next. Therefore the adjacent factors cancel, leaving only $\frac{1}{2} \cdot \frac{101}{100} = \frac{101}{200}$.

Three methods for solving recurrences

Method 1: Guess and check

Problem

If $a_n = 2a_{n-1} + 1$ *and* $a_0 = 0$, *find a formula for* a_n.

Three methods for solving recurrences

Method 1: Guess and check

Problem

If $a_n = 2a_{n-1} + 1$ and $a_0 = 0$, find a formula for a_n.

1. Find the first few values of a_n:

$$0,\ 1,\ 3,\ 7,\ 15,\ \ldots$$

2. Guess a formula: $a_n = 2^n - 1$.
3. We can prove this:

$$\begin{cases} 2^n - 1 = 2(2^{n-1} - 1) + 1, \\ 2^0 - 1 = 0. \end{cases}$$

Three methods for solving recurrences

Method 2: Convert to a sum

Problem

If $a_n = 2a_{n-1} + 1$ and $a_0 = 0$, find a formula for a_n.

1. Divide by a suitable function for step 2 to work.

$$\frac{a_n}{2^n} = \frac{a_{n-1}}{2^{n-1}} + \frac{1}{2^n}.$$

2. Make a substitution to write the recurrence as $s_n = s_{n-1} + \dots$.

$$s_n = s_{n-1} + \frac{1}{2^n} \qquad \text{where } s_n = \frac{a_n}{2^n}.$$

3. Solve for s_n and reverse the substitution.

$$s_n = 0 + \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} \quad \Rightarrow \quad a_n = 2^n s_n = 2^n - 1.$$

Three methods for solving recurrences

Method 3: Convert to a product

Problem

If $a_n = 2a_{n-1} + 1$ *and* $a_0 = 0$, *find a formula for* a_n.

1. Add or subtract something that makes step 2 work.

$$a_n + 1 = 2a_{n-1} + 2 = 2(a_{n-1} + 1).$$

2. Make a substitution to write the recurrence as $p_n = ? \cdot p_{n-1}$.

$$p_n = 2p_{n-1} \qquad \text{where } p_n = a_n + 1.$$

3. Solve for p_n and reverse the substitution.

$$p_n = 2^n p_0 = 2^n \quad \Rightarrow \quad a_n = p_n - 1 = 2^n - 1.$$

Exercises

1. Solve the recurrences:
   - $x_n = 2x_{n-1} + 2^n$, $x_0 = 0$.
   - $y_n = 3y_{n-1} + 2n - 1$, $y_0 = 0$.
   - $z_n = 3z_{n-1}^2$, $z_0 = 1$.
2. In preparation for the next topic, solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 1$.

Exercises

Solutions: # 1

We solve x_n by the method of sums. Dividing by 2^n, we get $\frac{x_n}{2^n} = \frac{x_{n-1}}{2^{n-1}} + 1$. This leads to the recurrence $s_n = s_{n-1} + 1$, with $s_0 = 0$, where $s_n = x_n/2^n$. This is easy to solve: $s_n = n$. Therefore $x_n = n \cdot 2^n$.

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We solve y_n by the method of products. Add $n+1$ to both sides and factor to get $y_n + n + 1 = 3(y_{n-1} + n)$. This means that for $p_n = y_n + n + 1$ we have the recurrence $p_0 = 1$ and $p_n = 3p_{n-1}$, so $p_n = 3^n$ and therefore $y_n = 3^n - n - 1$.

Exercises

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Finally, for z_n, we take the log of both sides. The recurrence $z_n = 3z_{n-1}^2$ becomes $\log_3 z_n = 2\log_3 z_{n-1} + 1$, with $\log_3 z_0 = 0$. This is the recurrence we took great pains to solve earlier, so $\log_3 z_n = 2^n - 1$, and therefore $z_n = 3^{2^n-1}$.

Exercises

Solutions: # 2

One way to approach the two-term recurrence is to begin with the method of products. Add a_{n-1} to both sides; then

$$a_n + a_{n-1} = 2a_{n-1} + 2a_{n-2} = 2(a_{n-1} + a_{n-2}).$$

If $p_n = a_n + a_{n-1}$, we have $p_n = 2p_{n-1}$, with $p_1 = 3$. Therefore $p_n = \frac{3}{2} \cdot 2^n$.

However, this doesn't solve the problem fully: a formula for p_n only tells us that

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Exercises

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However, this doesn't solve the problem fully: a formula for p_n only tells us that

$$a_n = -a_{n-1} + \frac{3}{2} \cdot 2^n.$$

We can solve this by the method of sums. Let $s_n = (-1)^n a_n$; then we have the recurrence $s_n = s_{n-1} + \frac{3}{2} \cdot (-2)^n$, with $s_0 = 2$. Therefore $s_n = 2 + \frac{3}{2} \sum_{k=1}^{n} (-2)^k = (-2)^n + 1$.

Finally, because $a_n = (-1)^n s_n$, we get $a_n = 2^n + (-1)^n$.

Solving linear recurrences

Recurrences such as $a_n = a_{n-1} + 2a_{n-2}$ come up so often there is a special method for dealing with these.

1. Find all solutions of the form $a_n = r^n$. We will need to solve a quadratic equation, since $r^n = r^{n-1} + 2r^{n-2}$ just means $r^2 = r + 2$. In this case, the solutions are $r = 2$ and $r = -1$.
2. Find a combination of these solutions that satisfies the initial conditions. Here, if $a_n = x \cdot 2^n + y \cdot (-1)^n$, we have

$$\begin{cases} x \cdot 2^0 + y \cdot (-1)^0 = a_0 = 2, \\ x \cdot 2^1 + y \cdot (-1)^1 = a_1 = 1. \end{cases}$$

So $a_n = 2^n + (-1)^n$.

In general, if a_n is a linear combination of $a_{n-1}, \ldots, a_{n-k}$, we'd have to solve a degree-k polynomial to get what we want.

Example of solving linear recurrences

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Find a formula for F_n.

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1. To find the exponential solutions, we solve $r^2 = r + 1$ to get $r = \dfrac{1 \pm \sqrt{5}}{2}$. Call these two solutions ϕ_1 and ϕ_2.
2. Now we know $F_n = x\phi_1^n + y\phi_2^n$. To find x and y, we solve:

$$\begin{cases} F_0 = x\phi_1^0 + y\phi_2^0, \\ F_1 = x\phi_1^1 + y\phi_2^1. \end{cases} \Leftrightarrow \begin{cases} x + y = 0, \\ \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) x + \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) y = 1. \end{cases}$$

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$$\begin{cases} F_0 = x\phi_1^0 + y\phi_2^0, \\ F_1 = x\phi_1^1 + y\phi_2^1. \end{cases} \Leftrightarrow \begin{cases} x + y = 0, \\ \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)x + \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)y = 1. \end{cases}$$

3. So $x = \frac{1}{\sqrt{5}}$, $y = -\frac{1}{\sqrt{5}}$, and

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

One thing that can go wrong

Consider the recurrence $a_n = 4a_{n-1} - 4a_{n-2}$, with $a_0 = 0$ and $a_1 = 2$. What goes wrong? Try to solve this using our previous methods.

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The problem is that $r^2 = 4r - 4$ only has one solution: $r = 2$. But multiples of 2^n aren't enough to fit the initial conditions.

Writing $a_n - 2a_{n-1} = 2(a_{n-1} - 2a_{n-2})$ yields the one-term recurrence $a_n = 2a_{n-1} + 2^n$. We have already solved this today; the solution is $a_n = n \cdot 2^n$.

In general, when a root r is repeated k times, it yields the solutions $a_n = r^n$, $a_n = n \cdot r^n$, ..., $a_n = n^{k-1} \cdot r^n$, and we solve for their combination as usual. (Here, all solutions to the recurrence are a combination of 2^n and $n \cdot 2^n$.)

Exercises

1. Solve the recurrence $x_n = 2x_{n-1} + x_{n-2}$ with $x_0 = x_1 = 1$.
2. Solve the recurrence $y_n = y_{n-1} + 2y_{n-2} + 2$ with $y_0 = 0$ and $y_1 = 1$.
3. Solve the recurrence $z_n = z_{n-1} + z_{n-2} - z_{n-3}$ with $z_0 = 1$, $z_1 = 0$, and $z_2 = 3$.

Exercises

Solutions

1. The characteristic equation is $r^2 = 2r + 1$, which yields $r = 1 \pm \sqrt{2}$. Determining the constants, we see that
$$x_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}.$$
2. By a trick analogous to the "method of products", we write $y_n + 1 = (y_{n-1} + 1) + 2(y_{n-2} + 1)$. Then $y_n + 1$ is some combination of 2^n and $(-1)^n$; from $y_0 + 1 = 1$ and $y_1 + 1 = 2$, we can deduce $y_n + 1 = 2^n$, so $y_n = 2^n - 1$.
3. Here we must first solve the cubic $r^3 = r^2 + r - 1$, whose three roots are $1, 1, -1$. This means z_n is a combination of 1, n, and $(-1)^n$. From the initial conditions, we see that $z_n = n + (-1)^n$.

More math problems

Inspired by 1990 VTRMC, #6. The sequence (y_n) obeys the recurrence $y_n = y_{n-1}(2 - y_{n-1})$. Solve for y_n in terms of y_0.

2005 AIME II, #11. For a positive integer m, let $a_0, a_1, \ldots, a_m$ be a sequence such that $a_0 = 37$, $a_1 = 72$, $a_{k+1} = a_{k-1} - \frac{3}{a_k}$ for $k = 1, \ldots, m-1$, and finally, $a_m = 0$. Find m.

2007 PU$\mathfrak{M}\alpha\mathbb{C}$, Algebra A #7. Two sequences x_n and y_n are defined by $x_0 = y_0 = 7$ and

$$\begin{cases} x_n = 4x_{n-1} + 3y_{n-1} \\ y_n = 3y_{n-1} + 2x_{n-1}. \end{cases}$$

Find $\lim_{n\to\infty} \frac{x_n}{y_n}$. *[Or just solve for x_n and y_n.]*

2011 VTRMC, #2. The sequence (a_n) is defined by $a_0 = -1$, $a_1 = 0$, and $a_n = a_{n-1}^2 - n^2 a_{n-2} - 1$. Find a_{100}.

More math problems

Solutions

1990 VTRMC, #6. Rewrite the recurrence as $1 - y_n = (1 - y_{n-1})^2$. It follows that $1 - y_n = (1 - y_0)^{2^n}$, so $y_n = 1 - (1 - y_0)^{2^n}$.

2005 AIME II, #11. Let $b_k = a_k a_{k+1}$. From the recurrence, we have $b_k = b_{k-1} - 3$, with $b_0 = 37 \cdot 72$. So $b_k = 37 \cdot 72 - 3k$. We have $b_{888} = a_{888} a_{889} = 0$; but $a_{888} \neq 0$ because $b_{887} = a_{887} a_{888} = 3$. Therefore $m = 889$.

2007 PU$\mathfrak{M}\alpha\mathbb{C}$, Algebra A #7. Note that $x_n - y_n = 2x_{n-1}$, so we can write x_n as $7x_{n-1} - 6x_{n-2}$. Solving this, we get $x_n = \frac{42}{5} \cdot 6^n - \frac{7}{5}$, so $y_n = \frac{28}{5} \cdot 6^n + \frac{7}{5}$. In the limit, $\frac{x_n}{y_n} \to \frac{42/5}{28/5} = \frac{3}{2}$.

2011 VTRMC, #2. From computing the first few terms, we guess that $a_n = n^2 - 1$, which is confirmed by induction:

$$n^2 - 1 = ((n-1)^2 - 1)^2 - n^2((n-2)^2 - 1) - 1.$$