Polynomials and Vieta's Formulas

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Review problems

1. If $a_0 = 0$ and $a_n = 3a_{n-1} + 2$, find a_{100}.

2. If $b_0 = 0$ and $b_n = n^2 - b_{n-1}$, find b_{100}.

Review problems

1. If $a_0 = 0$ and $a_n = 3a_{n-1} + 2$, find a_{100}.

   Since $a_n + 1 = 3a_{n-1} + 3 = 3(a_{n-1} + 1)$, we have $a_n + 1 = 3^n(a_0 + 1) = 3^n$, so $a_{100} = 3^{100} - 1$.

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2. If $b_0 = 0$ and $b_n = n^2 - b_{n-1}$, find b_{100}.

   Let $s_n = (-1)^n b_n$, so that $s_n = s_{n-1} + (-1)^n n^2$. With $s_0 = 0$, this means $s_n = \sum_{k=1}^{n} (-1)^k k^2$. Therefore

   $$b_{100} = s_{100} = 100^2 - 99^2 + 98^2 - \cdots + 2^2 - 1^2.$$

   To simplify this, observe that $k^2 - (k-1)^2 = k + (k-1)$, so

   $$b_{100} = 100 + 99 + 98 + \cdots + 2 + 1 = 5050.$$

Vieta's Formulas

The quadratic version

Suppose that $x = r_1$ and $x = r_2$ are the two solutions to the quadratic equation

$$x^2 + px + q = 0.$$

Then $r_1 \cdot r_2 = q$ and $r_1 + r_2 = -p$. To prove this, simply expand $(x - r_1)(x - r_2)$.

Slightly more generally, suppose that $x = r_1$ and $x = r_2$ are the two solutions of

$$ax^2 + bx + c = 0.$$

Then $r_1 \cdot r_2 = \frac{c}{a}$ and $r_1 + r_2 = -\frac{b}{a}$. This follows from the first version, if we divide the equation by a. For this reason, we assume the leading coefficient is 1 whenever we can.

Vieta's Formulas
Problems

Let a and b be the roots of $x^2 - 3x - 1 = 0$. Try to solve the problems below without finding a and b; it will be easier that way, anyway.

1. Find a quadratic equation whose roots are a^2 and b^2.
2. Compute $\frac{1}{a+1} + \frac{1}{b+1}$. (Hint: find a quadratic equation whose roots are $\frac{1}{a+1}$ and $\frac{1}{b+1}$ by manipulating the original.)

Surprise review problem:

3. Write a recurrence relation for the sequence $x_n = a^n + b^n$. (We can actually use this to compute something like $a^5 + b^5$ more or less painlessly.)

Vieta's Formulas

Solutions

1. We know $ab = -1$ and $a + b = 3$, and want to find a^2b^2 and $a^2 + b^2$. These are given by:

$$\begin{cases} a^2b^2 = (ab)^2 = (-1)^2 = 1 \\ a^2 + b^2 = (a+b)^2 - 2ab = 3^2 - 2(-1) = 11. \end{cases}$$

   Therefore $x^2 - 11x + 1 = 0$ is the equation we want.

   Another approach: expand $(x^2 - 3x - 1)(x^2 + 3x - 1)$ to get $x^4 - 11x^2 + 1$, then substitute x for x^2. Why does this work?

2. The equation $(x-1)^2 - 3(x-1) - 1 = 0$, or $x^2 - 5x + 3 = 0$, has roots $a + 1$ and $b + 1$. Similarly, $(1/x)^2 - 5(1/x) + 3 = 0$, or $1 - 5x + 3x^2 = 0$, has roots $\frac{1}{a+1}$ and $\frac{1}{b+1}$. Therefore $\frac{1}{a+1} + \frac{1}{b+1} = \frac{5}{3}$.

3. The recurrence is $x_n = 3x_{n-1} + x_{n-2}$, with $x_0 = 2$ and $x_1 = 3$.

Vieta's Formulas

The general version

In general, given the equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0,$$

we know the following:

- The sum of the roots (with multiplicity) is $-a_{n-1}$.
- The product of the roots is $(-1)^n a_0$.
- In general, $(-1)^k a_{n-k}$ is the sum of all k-fold products of the roots; for example, in a cubic equation with roots r_1, r_2, r_3, the coefficient of x is $r_1 r_2 + r_1 r_3 + r_2 r_3$.

(Don't forget that if x^n has a coefficient too, you should divide by it before applying these rules.)

Vieta's Formulas
Problems

1. (HMMT 1998) Three of the roots of $x^4 + ax^2 + bx + c = 0$ are 2, -3, and 5. Find the value of $a + b + c$.
2. (AIME 2001) Find the sum of all the roots of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$.
3. (AIME 1996) Suppose the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$.
    - By way of a warm-up, find r.
    - The real problem is to find t.

Vieta's Formulas

Solutions

1. The coefficient of x^3 is 0, so the sum of the roots is 0, and the fourth root must be -4. The polynomial factors as $(x-2)(x-5)(x+3)(x+4)$. Setting $x=1$ gives $1+a+b+c=(1-2)(1-5)(1+3)(1+4)=80$, so $a+b+c=79$.

2. Using Vieta's formulas is left as an exercise. A shortcut solution is to observe that if x is a root, so is $\frac{1}{2}-x$, and each such pair yields a total of $\frac{1}{2}$. There are 2000 roots, so the total is 500.

3.
   - We have $r=-((a+b)+(b+c)+(a+c))=$
     $=-2(a+b+c)=6$.
   - To evaluate $t=-(a+b)(a+c)(b+c)$, it helps to observe $a+b+c=-3$ and write $a+b=-3-c$ and so on. Then $t=-P(-3)=23$.

More problems

Let α, β, γ be the roots of $x^3 - 3x^2 + 1$.

1. Find a polynomial whose roots are $\alpha + 3$, $\beta + 3$, and $\gamma + 3$.
2. Find a polynomial whose roots are $\frac{1}{\alpha+3}$, $\frac{1}{\beta+3}$, and $\frac{1}{\gamma+3}$.
3. Compute $\frac{1}{\alpha+3} + \frac{1}{\beta+3} + \frac{1}{\gamma+3}$.
4. Find a polynomial whose roots are α^2, β^2, and γ^2.
5. Find a recurrence relation for $x_n = \alpha^n + \beta^n + \gamma^n$, and use it to compute $\alpha^5 + \beta^5 + \gamma^5$.

More problems

Solutions

1. To increment all roots by 3, substitute $x - 3$ for x. This yields $x^3 - 12x^2 + 45x - 53$.
2. Reversing the coefficients to get $1 - 12x + 45x^2 - 53x^3$ yields a polynomial whose roots are reciprocals of the polynomial above. (Do you see why?)
3. This is just the sum of the roots of the polynomial above: $\frac{45}{53}$.
4. Observe that $x^3 + 3x^2 - 1$ has roots $-\alpha$, $-\beta$, and $-\gamma$. Therefore $(x^3 - 3x^2 + 1)(x^3 + 3x^2 - 1)$ has roots $\pm\alpha, \pm\beta, \pm\gamma$ and factors as $(x^2 - \alpha^2)(x^2 - \beta^2)(x^2 - \gamma^2)$. Replacing x^2 by x yields our answer: $x^3 - 9x^2 + 6x - 1$.
5. The recurrence is $x_n = 3x_{n-1} - x_{n-3}$. We have $x_0 = 3$, $x_1 = 3$, and $x_2 = 9$, so $x_3 = 24$, $x_4 = 69$, and $x_5 = 198$.

Even more problems

1. (Canada 1988) For some integer a, the equations $1988x^2 + ax + 8891 = 0$ and $8891x^2 + ax + 1988 = 0$ share a common root. Find a.
2. (ARML 1989) If $P(x)$ is a polynomial in x such that for all x,

   $$x^{23}+23x^{17}-18x^{16}-24x^{15}+108x^{14} = (x^4-3x^2-2x+9)\cdot P(x),$$

   compute the sum of the coefficients of $P(x)$.

Even more problems

Solutions

1. Let x be the common root; then by subtracting the two equations, we have

$$(8891 - 1988)x^2 + (1988 - 8891) = 0$$

so $x^2 - 1 = 0$, and therefore $x = \pm 1$. Plug ± 1 into one of the equations to get $1988 \pm a + 8891 = 0$ and therefore $a = \pm 10879$.

2. The sum of the coefficients of $P(x)$ is $P(1)$. Setting $x = 1$, we get

$$1 + 23 - 18 - 24 + 108 = (1 - 3 - 2 + 9)P(1).$$

Solving, we obtain $P(1) = 18$.