

Series Solutions

Joseph Zoller

February 14, 2016

Solutions

1. (1980 ARML Team Round Problem 6) In an arithmetic progression, the ratio of the sum of the first r terms to the sum of the first s terms is equal to the ratio of r^2 to s^2 ($r \neq s$). Compute the ratio of the 8th term to the 23rd term.

[Solution:]

2. (1992 ARML Individual Round Problem 4) Consider the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots$, where the integer n appears n times. Compute the 1992nd term of this sequence.

[Solution: 63]

Find the smallest $n \in \mathbb{N}$ such that

$$\sum_{i=1}^n i \geq 1992 \implies n(n+1) \geq 2(1992) = 3984 = 64^2 - 112 = (63)(64) - 48 \implies n = 63$$

Thus, the 1992nd term of this sequence is a $\boxed{63}$.

3. (1993 ARML Team Round Problem 8) The arithmetic mean of the positive numbers a_1, a_2, \dots, a_k is one-fourth of the arithmetic mean of the positive numbers b_1, b_2, \dots, b_{7k} (where k is a positive integer). If both of these means are integers, compute the smallest possible integer value for the arithmetic mean of

$$a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_{7k}$$

[Solution: 29, when the means are 8 and 32]

First, $a_1 + a_2 + \dots + a_k = kS$ and $b_1 + b_2 + \dots + b_{7k} = (7k)(4S) = 28kS$. Thus,

$$\implies \frac{a_1 + a_2 + \dots + a_k + b_1 + b_2 + \dots + b_{7k} = 29kS}{8k} = \frac{29kS}{8k} = \frac{29S}{8} = 29 \left(\frac{S}{8} \right)$$

Thus, the smallest value is when $S = 8$, and the mean is $\boxed{29}$.

4. (1993 ARML Team Round Problem 10) (*Reminder:* The Fibonacci sequence is defined as follows: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.)

The Fibonacci numbers F_a , F_b , and F_c form an arithmetic sequence. If $a + b + c = 2000$, compute a .

[Solution:]

5. (1999 ARML Team Round Problem 7) Define a sequence of integers as follows: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 5$, $a_5 = 7$, $a_6 = 9$, the next four terms are the next four even integers after 9, the next five terms are the next five odd integers after 16, and so on. Compute a_{1999} .

[Solution:]

From the pattern, it can be observed that for every $n \in \mathbb{N}$, $a_{n(n+1)/2} = n^2$. Also, note that

$$\frac{(62)(62+1)}{2} = 1953 \implies a_{1953} = 62^2, \quad \frac{(63)(63+1)}{2} = 2016 \implies a_{2016} = 63^2$$

Thus, all the terms between a_{1953} and a_{2016} are going to be odd, so

$$a_{1999} = a_{2016-17} = a_{2016} - 17(2) = 63^2 - 34 = 3969 - 34 = 3935$$

6. (2006 AMC 12A Problem 12) A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?

[Solution: 173 cm]

7. (2003 AIME I Problem 2) One hundred concentric circles with radii $1, 2, 3, \dots, 100$ are drawn in a plane. The interior of the circle of radius 1 is colored red, and each region bounded by consecutive circles is colored either red or green, with no two adjacent regions the same color. The ratio of the total area of the green regions to the area of the circle of radius 100 can be expressed as m/n , where m and n are relatively prime positive integers. Find $m + n$.

[Solution: $m + n = 301$, where $m = 101$ and $n = 200$]

To get the green area, we can color all the circles of radius 100 or below green, then color all those with radius 99 or below red, then color all those with radius 98 or below green, and so forth. This amounts to adding the area of the circle of radius 100, but subtracting the circle of radius 99, then adding the circle of radius 98, and so forth.

The total green area is thus given by $100^2\pi - 99^2\pi + 98^2\pi - 97^2\pi + \dots + 2^2\pi - 1^2\pi$, while the total area is given by $100^2\pi$, so the ratio is

$$\begin{aligned} \frac{100^2\pi - 99^2\pi + 98^2\pi - 97^2\pi + \dots + 2^2\pi - 1^2\pi}{100^2\pi} &= \frac{(100^2 - 99^2) + (98^2 - 97^2) + \dots + (2^2 - 1^2)}{100^2} \\ &= \frac{(100 + 99) + (98 + 97) + \dots + (2 + 1)}{100^2} = \frac{1}{100^2} \cdot \left(\frac{100(1 + 100)}{2} \right) = \frac{101}{200} \\ &\implies m = 101, n = 200 \implies m + n = \boxed{301} \end{aligned}$$

8. (2012 AIME I Problem 2) The terms of an arithmetic sequence add to 715. Then, a new sequence is constructed by doing the following: The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the k th term is increased by the k th odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

[Solution: 195]

If the sum of the original sequence is $\sum_{i=1}^n a_i = 715$ then the sum of the new sequence can be expressed as $\sum_{i=1}^n [a_i + (2i - 1)] = n^2 + \sum_{i=1}^n a_i = 836$. Therefore,

$$836 = n^2 + 715 \implies n^2 = 836 - 715 = 121 \implies n = 11$$

Now the middle term of the original sequence is simply the average of all the terms, or $\frac{715}{11} = 65$, and the first and last terms average to this middle term, so the desired sum is simply three times the middle term, or $\boxed{195}$.

9. (2005 AIME II Problem 3) An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is m/n where m and n are relatively prime integers. Find $m + n$.

[Solution: $m + n = 802$, where $m = 399$ and $n = 403$]

Let's call the first term of the original geometric series a and the common ratio r , so $2005 = a + ar + ar^2 + \dots$. Using the sum formula for infinite geometric series, we have

$$\frac{a}{1-r} = 2005$$

Then we form a new series, $20050 = a^2 + a^2r^2 + a^2r^4 + \dots$. We know this series has sum

$$\frac{a^2}{1-r^2} = 20050 \implies \frac{a}{1-r} \cdot \frac{a}{1+r} = 2005 \cdot \frac{a}{1+r} = 20050 \implies \frac{a}{1+r} = 10$$

Then,

$$\begin{aligned} a = 2005 - 2005r = 10 + 10r &\implies 1995 = 2015r \implies r = \frac{1995}{2015} = \frac{399}{403} \\ &\implies m = 399, n = 403 \implies m + n = \boxed{802} \end{aligned}$$

10. (2007 AIME II Problem 12) The increasing geometric sequence x_0, x_1, x_2, \dots consists entirely of integral powers of 3. Given that

$$\sum_{n=0}^7 \log_3(x_n) = 308, \quad 56 \leq \log_3\left(\sum_{n=0}^7 x_n\right) \leq 57,$$

find $\log_3(x_{14})$.

[Solution: $\log_3(x_{14}) = 91$, where $x_0 = 3^{21}$ and $r = 3^5$]

Suppose that $x_0 = a$, and that the common ratio between the terms is r .

The first conditions tells us that:

$$\log_3 a + \log_3 ar + \dots + \log_3 ar^7 = 308 \implies \log_3 a^8 r^{1+2+\dots+7} = 308 \implies a^8 r^{28} = 3^{308}$$

Since all of the terms of the geometric sequence are integral powers of 3, we know that both a and r must be powers of 3. Denote $3^x = a$ and $3^y = r$. We find that $8x + 28y = 308$. The possible positive integral pairs of (x, y) are $(35, 1)$, $(28, 3)$, $(21, 5)$, $(14, 7)$, $(7, 9)$, $(0, 11)$.

The second condition tells us that:

$$56 \leq \log_3(a + ar + \dots + ar^7) \leq 57 \implies 3^{56} \leq 3^x \frac{3^{8y} - 1}{3^y - 1} \leq 3^{57} \implies 3^{56-x} \leq \frac{3^{8y} - 1}{3^y - 1} \leq 3^{57-x}$$

The fractional part $\frac{3^{8y}-1}{3^y-1} \approx \frac{3^{8y}}{3^y} = 3^{7y}$. Thus, we need approximately

$$56 - x \leq 7y \leq 57 - x$$

Checking the pairs above, only $(x, y) = (21, 5)$ is close.

Our solution is therefore $\log_3(x_{14}) = \log_3(ar^{14}) = \log_3 3^x + \log_3 3^{14y} = x + 14y = \boxed{91}$