Algebraic Manipulations

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In most school math classes, equations and systems of equations are meant to be solved. After all, if we could pinpoint down the exact values of the variables that can be plugged in to make the equality true, then we have in effect completely understood the equation at hand. However, sometimes these values are infeasible to compute by hand and sometimes they aren't even important to what you're trying to do! In these situations, algebraic finesse is crucial to success.

Throughout this packet, you will see examples of problems where taking advantage of symmetry is key. This is because symmetry is nice enough that it often helps with exploiting patterns - and patterns are often the driving force behind making complicated problems simple.

1 Examples

1. [AMSP Team Contest 2013] Positive real numbers a and b satisfy

$$a + \frac{1}{b} = 5 \quad \text{and} \quad b + \frac{1}{a} = 7.$$

Compute $ab + \frac{1}{ab}$.

2. Suppose x and y are real numbers such that $x + y = 3$ and $xy = 1$. Compute $x^4 - 5x^3y - 5xy^3 + y^4$.

3. [Math League HS 1997-1998/2009-2010] In terms of x, what is the polynomial P of least degree, with integral coefficients, for which $P(\sqrt{3} + \sqrt{2}) = \sqrt{3} - \sqrt{2}$?

4. [MATHCOUNTS 2009] If $x + \frac{1}{y} = 1$ and $y + \frac{1}{z} = 1$, what is xyz?

5. Factor $a^3 + b^3 + c^3 - 3abc$.

6. [CMIMC 2017, Own] Suppose x, y, and z are nonzero complex numbers such that $(x + y + z)(x^2 + y^2 + z^2) = x^3 + y^3 + z^3$. Compute

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

2 Problems

Throughout this set, try to solve the problems without adding any extra assumptions, e.g. finding a specific case of variables for which the problem works out. No problem here will require this.

Problems are ordered roughly in terms of difficulty. Challenging problems are marked with a ★.

1. Two warm-up problems.

   (a) [Math League HS 1991-1992] If $x + y = 5$ and $x - y = 1$, what is the value of $2^{x^2-y^2}$?

   (b) [Math League HS 2008-2009] If $(x - 1)(y - 1) = 2008$, what is the value of $(1 - x)(1 - y)$?

2. Let p and q be positive real numbers such that $p + q = 7$ and $pq = 5$. Compute

   i) $p^2q + pq^2$

   ii) $p(1 - q) + q(1 - p)$

   iii) $p^3 + q^3$

   iv) $\dfrac{p}{q - 1} + \dfrac{q}{p - 1}$

3. Suppose $x > 1$ is a real number such that $x + \frac{1}{x} = \sqrt{22}$. What is
   (a) $x^2 + \frac{1}{x^2}$?
   (b) [CMIMC 2018, Own] $x^2 - \frac{1}{x^2}$?

4. [NIMO 16, Own] Let a and b be positive real numbers such that $ab = 2$ and
$$\frac{a}{a+b^2} + \frac{b}{b+a^2} = \frac{7}{8}.$$
Find $a^6 + b^6$.

5. [Math League HS 1998-1999] Suppose a and b are real numbers such that
$$\frac{1}{a(b+1)} + \frac{1}{b(a+1)} = \frac{1}{(a+1)(b+1)}.$$
What is $\frac{1}{a} + \frac{1}{b}$?

6. [CMIMC 2016, Own] Let ℓ be a real number satisfying the equation $\frac{(1+\ell)^2}{1+\ell^2} = \frac{13}{37}$. Then
$$\frac{(1+\ell)^3}{1+\ell^3} = \frac{m}{n},$$
where m and n are positive coprime integers. Find $m + n$.

7. [AMC10 2000] Two non-zero real numbers, a and b, satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?
   (A) -2 **(B)** $-\frac{1}{2}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** 2

8. [Mandelbrot 2013-2014] Let c be the larger solution to the equation $x^2 - 20x + 13 = 0$. Compute the area of the circle with center (c, c) passing through the point $(13, 7)$.

9. Suppose a, b, and c are real numbers such that
$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right).$$
If $abc = 13$, what is $a + b + c$?

10. Two similar-looking Math League problems.
    (a) [Math League HS 1982-1983] What are all ordered pairs of numbers (x, y) which satisfy
$$x^2 - xy + y^2 = 7 \qquad \text{and} \qquad x - xy + y = -1?$$
    (b) [Math League HS 1990-1991] What is the ordered pair of numbers (x, y), with $x > y$, for which
$$x^2 + xy + y^2 = 84 \qquad \text{and} \qquad x + \sqrt{xy} + y = 14?$$

11. [NIMO 20, Michael Tang] Suppose a, b, c, and d are positive real numbers which satisfy the system of equations
$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= 762, \\ ab + cd &= 260, \\ ac + bd &= 365, \\ ad + bc &= 244. \end{aligned}$$
Compute $abcd$.

12. [Math Prize 2012] Evaluate the expression

$$\frac{121\left(\frac{1}{13}-\frac{1}{17}\right)+169\left(\frac{1}{17}-\frac{1}{11}\right)+289\left(\frac{1}{11}-\frac{1}{13}\right)}{11\left(\frac{1}{13}-\frac{1}{17}\right)+13\left(\frac{1}{17}-\frac{1}{11}\right)+17\left(\frac{1}{11}-\frac{1}{13}\right)}.$$

★ 13. [AMSP Team Contest] Let a, b, c be nonzero numbers such that $a^2-b^2=bc$ and $b^2-c^2=ac$. Prove that $a^2-c^2=ab$.

★ 14. [CMIMC 2017, Own] Let a, b, and c be complex numbers satisfying the system of equations

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=9,$$
$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=32,$$
$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}=122.$$

Find abc.

★ 15. [USAMO 1989] Let u and v be real numbers such that

$$(u+u^2+u^3+\cdots+u^8)+10u^9=(v+v^2+v^3+\cdots+v^{10})+10v^{11}=8.$$

Determine, with proof, which of the two numbers, u or v, is larger.