Systems of Equations

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1 Problems

1. (07 AMC 10 # 9) Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. Compute ab.

2. (01 AMC 10 #10) If x, y, and z are positive with $xy = 24$, $xz = 48$, and $yz = 72$, then compute $x + y + z$.

3. (90 ARML Individual #3) If $a + b = c$, $b + c = d$, $c + d = a$, and b is a positive integer, compute the greatest possible value for $a + b + c + d$.

4. (NYCIML F06B08) Given that $x+y+z = 7$ and $x^2+y^2+z^2 = 10$, compute $xy+yz+xz$.

5. (99 ARML Team #2) Compute the number of ordered triples of integers (x, y, z), $1729 < x, y, z, < 2017$ which satisfy:
$$x^2 + xy + y^2 = y^3 - x^3 \quad \text{and} \quad yz + 1 = y^2 + z$$

6. (87 ARML Team #9) If a, b, c, x, y, and z are real and $a^2+b^2+c^2 = 25$, $x^2+y^2+z^2 = 36$, and $ax + by + cz = 30$, compute
$$\frac{a + b + c}{x + y + z}.$$

7. (00 AMC 12 #20) Let x, y, z be positive numbers satisfying
$$x + \frac{1}{y} = 4, \quad y + \frac{1}{z} = 1, \quad z + \frac{1}{x} = \frac{7}{3}$$
Compute xyz.

8. (00 AIME I #9)The system of equations
$$\begin{aligned} \log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4 \\ \log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1 \\ \log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0 \end{aligned}$$
has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$

9. (NYCIML F06 B24) Compute all ordered triples of integers (x, y, z) satisfying
$$\begin{aligned} x + yz &= 6 \\ y + xz &= 6 \\ z + xy &= 6 \end{aligned}$$

10. (02 AMC 10B #20) Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Compute $a^2 - b^2 + c^2$.

11. (02 AIME I #6) The solutions to the system of equations

$$\begin{aligned} \log_{225} x + \log_{64} y &= 4 \\ \log_x 225 - \log_y 64 &= 1 \end{aligned}$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1y_1x_2y_2)$.

2 Challenge Problems

1. (01 Putnam B2) Find all pairs of real numbers (x, y) satisfying the system of equations

$$\begin{aligned} &\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2) \\ &\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4). \end{aligned}$$

2. (93 USAMO #1) For each integer $n \geq 2$, determine, with proof, which of the two positive real numbers a and b satisfying

$$a^n = a + 1, \qquad b^{2n} = b + 3a$$

is larger.

3 Background

The *logarithm* function is defined by $\log_a b$ equals the number c such that $a^c = b$ (this is only defined when $a > 0$ and $a \neq 1$). For example, $\log_2 8 = 3$ because $2^3 = 8$ and $\log_2 \frac{1}{4} = -2$ because $2^{-2} = \frac{1}{4}$. The basic properties of log are:

- $\log_a(bc) = \log_a b + \log_a c$.
- $\log_a(b^c) = c \log_a b$.
- $\log_a b = \frac{\log_c b}{\log_c a}$ for any $c > 0$ such that $c \neq 1$.

4 Solutions

Solutions to AMC, AIME, and NYCIML problems can be found online, see the links on the archive page. To find the solutions for NYCIML problems, the code F06B08 e.g. indicates that this was in the Fall 2006 competition, from the Senior B test, problem #08. The solutions are found at the bottom of the corresponding document on the NYCIML site.

The solutions to ARML problems cannot be found online, so we have listed them here.

(2.) Substitute $a + b = c$ into $b + c = d$ to get $a + 2b = d$. Substitute $a + b = c$ and $a + 2b = d$ into $c + d = a$ to get $2a + 3b = a$, which gives $a = -3b$. Substituting back into $a + b = c$ and $a + 2b = d$ gives $c = -2b$ and $d = -b$, so $a + b + c + d = -3b + b - 2b - b = -5b$. This is greatest when b is as small as possible, and since b is a positive integer, that is when $b = 1$, which gives $a + b + c + d = -5$.

(5.) Factor the right side of the first equation into $y^3 - x^3 = (y-x)(x^2+xy+y^2)$. Since x, y are positive, $x^2+xy+y^2 > 0$ and we may cancel it from both sides of the equation, giving $y - x = 1$, or $y = x+1$. Rearrange the second equality to $yz - z = y^2 - 1$, and factor to $z(y-1) = (y+1)(y-1)$. Since $y > 1$, we can cancel the $y-1$ terms to get $z = y+1$. If x is any integer between 1730 and 2014 inclusive, then x, y, z all fall between 1729 and 2017. There are $2014 - 1730 + 1 = 285$ such possibilities.

(6.) The easiest way to solve this is to realize $(a, b, c) = (5, 0, 0)$ and $(x, y, z) = (6, 0, 0)$ is a valid solution, from which $\frac{a+b+c}{x+y+z} = \frac{5}{6}$. To prove that this is the only possibility requires the Cauchy-Schwarz inequality, which is beyond the scope of these solutions.