Complex Numbers

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Problems

If you are unfamiliar with any of the words or symbols below, please refer to the next page for a collection of the relevant information to help you!

1. Compute $|1+2i|^2$ and $(1+2i)^2$. Do the same for $|2+3i|^2, (2+3i)^2$. Do you notice anything special about the numbers you find?
2. (F06 NYCIML B19) Compute $(1-i)^{10}$.
3. (F11 NYCIML B2) Let z be a complex number that satisfies $z+6i = iz$. Find z.
4. (04 AMC 12B #16) A function f is defined by $f(z) = i\overline{z}$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?
5. (17 AMC 12A #17) There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?
6. Using de Moivre's theorem, find formulas for $\cos 2\theta$, $\cos 3\theta$ in terms of $\cos\theta$.
7. (09 AMC 12A #15) For what value of n is $i + 2i^2 + 3i^3 + \cdots + ni^n = 48 + 49i$?
8. (F09 NYCIML B23) Find the complex number c such that the equation $x^2+4x+6ix+c = 0$ has only one solution.
9. (09 AIME I #2) There is a complex number z with imaginary part 164 and a positive integer n such that $\frac{z}{z+n} = 4i$. Find n.
10. (85 AIME #3) Find c if a, b, c are positive integers which satisfy $c = (a+bi)^3 - 107i$. (Hint: 107 is prime.)
11. (94 AIME #8) The points $(0,0), (a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab. (Hint: is there a way to think of points in the plane as complex numbers?)
12. (97 AIME #14) Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$.

Challenge Problems

1. If p is a prime number and $a_0, a_1, \dots, a_{p-1}$ are rational numbers satisfying

$$a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{p-1}\zeta^{p-1} = 0,$$

2. (95 IMO) Let $p > 2$ be a prime number and let $A = \{1, 2, \dots, 2p\}$. Find the number of subsets of A each having p elements and whose sum is divisible by p. (Hint: Count the more general N_k, the number of subsets of A with p elements whose sum is congruent to $k \bmod p$ and consider p^{th} roots of unity.)

Background

A *complex number* is a number of the form $z = a + bi$, where a, b are real numbers and i is the number satisfying $i^2 = -1$. a is referred to as the *real part* of z, denoted by $\operatorname{Re} z$, and b is referred to as the *imaginary part*, denoted by $\operatorname{Im} z$. Addition and multiplication of imaginary numbers works the same way as multiplication of binomials; that is, if $z = a + bi, w = c + di$, where $a, b, c, d \in \mathbb{R}$ then

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$\text{w } zw = (a + bi)(c + di) = ac + bdi^2 + i(ad + bc) = (ac - bd) + i(ad + bc).$$

For a complex number $z = a + bi$, we define

- the *conjugate* of a z, denoted $\overline{z}$, is $a - bi$;
- the *norm* or *modulus* of z, denoted $|z|$, is $\sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$.

de Moivre's formula

One useful fact is that any complex number z can be written in the form

$$z = |z| \operatorname{cis} \theta,$$

where $0 \le \theta < 2\pi$, and $\operatorname{cis} \theta = \cos \theta + i \sin \theta$. Often, θ is referred to as the *argument* of z. Then, de Moivre's formula states that for any $n \in \mathbb{N}$, $z^n = |z|^n \operatorname{cis}(n\theta)$. This provides an important connection between complex numbers and trigonometry.