Sequences and Series

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1 Problems

1. Let a_n be a sequence defined by $a_1 = 0$, $a_n = a_{n-1} + 3$ for $n \geq 2$ and let $S_n = \sum_{k=1}^{n} a_k$. Find S_{2017}.

2. How many geometric sequences with integer ratios have $a_1 = 3$ and $a_n = 12288$ for some n?

3. The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term? [2]

4. Compute $\sum_{n=1}^{\infty} \frac{2^n-1}{3^{n-1}}$.[3]

5. Let $a_1, a_2, \ldots, a_k$ be a finite arithmetic sequence with $a_4 + a_7 + a_{10} = 17$ and $\sum_{n=4}^{14} a_n = 77$. If $a_k = 13$, what is k?[4]

6. Let a_n be a sequence with $a_1 = 1, a_2 = 3$ and $a_n = 2a_{n-1} + a_{n-2}$. Find the remainder when a_{2017} is divided by 4.

7. Let $a_0 = a_1 = 1$, $a_{n+1} = a_n a_{n-1} + 1$. Show that 4 is not a divisor of a_{2017}.[5]

8. Let $a_1 = a_2 = 1, a_3 = -1$ and $a_n = a_{n-1}a_{n-3}$ for $n \geq 4$. Find a_{2017}. [5]

9. Find the value of $a_2 + a_4 + \cdots + a_{98}$ if a_n is an arithmetic progression with common difference 1 and $a_1 + a_2 + \cdots + a_{98} = 137$. [6]

10. Let $a_0 = 1, a_1 = 3$ and $a_n = \frac{a_{n-1}^2+1}{2}$ for $n > 2$. Let $S_n = \frac{1}{a_n - 1} + \sum_{k=0}^{n-1} \frac{1}{a_k+1}$. Find S_{2017}.

11. A sequence is defined as follows $a_1 = a_2 = a_3 = 1$, and, for all positive integers $n, a_{n+3} = a_{n+2} + a_{n+1} + a_n$. Given that $a_{28} = 6090307, a_{29} = 11201821$, and $a_{30} = 20603361$, find the remainder when $\sum_{k=1}^{28} a_k$ is divided by 1000.[7]

12. Let $a_1 = a_2 = 1$ and $a_n = \frac{a_{n-1}^2+2}{a_{n-2}}$ for $n \geq 3$. Show that every a_i is an integer. [5]

[1] Many thanks to David Altizio and Elliot Haney for their help in compiling problems!

[2] AMC 12A 2014 #7

[3] NYCIML S11A3

[4] AHSME 1993 #21

[5] Arthur Engel's *Problem Solving Strategies*

[6] AIME 1984 #1

[7] AIME II 2006 #11

2 Challenge Problems

1. The real numbers $a_0, a_1, \dots, a_{2013}$ and $b_0, b_1, \dots, b_{2013}$ satisfy $a_n = \frac{1}{63}\sqrt{2n+2} + a_{n-1}$ and $b_n = \frac{1}{96}\sqrt{2n+2} - b_{n-1}$ for every integer $n = 1, 2, \dots, 2013$. If $a_0 = b_{2013}$ and $b_0 = a_{2013}$, compute $\sum_{k=1}^{2013}(a_k b_{k-1} - a_{k-1}b_k)$. [8]

2. Start with two positive integers x_1, x_2, both less than 10000, and for $k \geq 3$ let x_k be the smallest of the absolute values of the pairwise differences of the preceding terms. Prove that we always have $x_{21} = 0$. [9]

3 Background

A sequence is an enumerated/ordered collection of terms which can be finite/infinite. A series is formed by taking cumulative partial sums of a sequence. For a sequence of terms $a_1, a_2, \dots, a_n$, we denote $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

Remark 1. Sequences are often defined by recurrences which give a start term x_0 and/or x_1 and then some relation between x_n and previous terms. Sometimes they can also be described by a closed formula.

Example 2 (Arithmetic Sequence/Series)**.** Given a sequence starting at a_1 with $a_{n+1} = a_n + d$ for some fixed d, then $a_{n+1} = a_1 + nd$ and $\sum_{i=1}^{n} a_i = \frac{n(2a_1+(n-1)d)}{2}$

Example 3 (Geometric Sequence/Series)**.** Given a sequence starting at b_1 with $b_{n+1} = a_n r$ for some fixed r, then $a_{n+1} = a_1 r^n$ and $\sum_{i=1}^{n} a_i = \frac{a_1(1-r^n)}{1-r}$. If $|r| < 1$, we have that the infinite geometric series has sum $\frac{a}{1-r}$.

Remark 4. A pretty good way to approach a lot of sequence/series problems is to write out a few (or many, if you so desire) terms and look for a pattern.

Some series may "telescope" if you rewrite them the right way:

Example 5. Evaluate the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{99\cdot 100}.$$

Proof. If we rewrite this as $\frac{1}{2} = 1 - \frac{1}{2}$, $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$, $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$, and so on, then the sum becomes

$$\left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{99}} - \frac{1}{100}\right) = 1 - \frac{1}{100} = \boxed{\frac{99}{100}}$$

□

[8] Fall OMO 2013
[9] AUO 1976

4 Selected Solutions

For solutions to AMC, AHSME, AIME, NYCIML problems, see the links on the Archive page.

1. a_n defines an arithmetic sequence. By the formula in example 2, $\sum_{i=1}^{2017} = \frac{2017(2\cdot 0+(2016)3}{2} = 6099408$.

6. There are only $4 \cdot 4$ possible values of $(a_n, a_{n+1}) \pmod 4$. Since a_{n+2} depends only on a_n and a_{n+1}, the sequence will eventually repeat itself.

   Computing the first few values of $a_n \pmod 4$ we have $a_0 \equiv 1$, $a_1 \equiv 3$, $a_2 \equiv 3$, $a_3 \equiv 1$, $a_4 \cong 1$, $a_5 \cong 3$. Since $(a_0, a_1) = (a_4, a_5)$, this pattern will repeat with period 4. $2017 \equiv 1 \pmod 4$ so $a_{2017} \equiv a_1 \equiv 3$, so the remainder is 3.

7. Similarly to problem 6, there are only $4 \cdot 4$ possible values of $(a_n, a_{n+1}) \pmod 4$. Since a_{n+2} depends only on a_n and a_{n+1}, the sequence will eventually repeat itself.

   $a_0 \equiv 1$, $a_1 \equiv 1$, $a_2 \equiv 3$, $a_3 \equiv 0$, $a_4 \equiv 1$, $a_5 \equiv 1$. Since $(a_0, a_1) = (a_4, a_5)$, this pattern will repeat with period 4. $2017 \equiv 1 \pmod 4$ so $a_{2017} \equiv a_1 \equiv 1$. Since $a_{2017} \not\equiv 0 \pmod 4$, a_{2017} is not divisible by 4.

8. Similar to questions 6 and 7–the repeating pattern is $1, 1, -1, -1, -1, 1, -1, 1, 1, -1, -1, -1, 1, -1, \ldots$. The period of this sequences is 7 and $2017 \cong 1 \pmod 7$ so the answer is the same as a_1, which is 1.

10. Using partial fractions,

$$\frac{a}{a_{k+1}-1} = \frac{1}{a_k - 1} + \frac{1}{a_k + 1}$$

Rearrange to get

$$\frac{1}{a_k+1} = \frac{1}{a_k - 1} - \frac{a}{a_{k+1} - 1}$$

Plug this in for each term of the sum to get

$$S_n = \frac{1}{a_n - 1} + \frac{a}{a_0 - 1} + \sum_{k=1}^{n-1} \frac{1}{a_k - 1} - \frac{1}{a_{k+1} - 1}$$

The summation telescopes leaving

$$S_n = \frac{1}{a_n - 1} + \frac{a}{a_0 - 1} + \frac{1}{a_n - 1} - \frac{a}{a_n - 1} = \frac{1}{2} + \frac{1}{2} = 1$$