Sequences and Series

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1 Varsity Warm-up

1. (Stirling Numbers of the Second Kind) Find a recursive formula for the number of ways to divide n distinguishable objects into m indistinguishable boxes, where no box is allowed to be empty.

2. Consider the classic Tower of Hanoi problem, where there are 3 rods, and a stack of n disks of ascending order of size on one rod. You want to move the entire stack to another rod, with the following rules: Only one disk can be moved at a time, each move consists of taking the top disk from one rod and placing it at the top of another rod, and larger disks cannot go on top of smaller disks. Determine a recurrence that describes the minimum number of moves required to solve such a puzzle, then solve your recurrence.

3. Solve the recurrence $a_n = 2a_{n-1} + 3a_{n-2}$ where $a_0 = 2$ and $a_1 = 6$. If you were to get a general linear recurrence such as $a_n = Ba_{n-1} + Ca_{n-2}$, with initial conditions $a_0 = x$ and $a_1 = y$, how could you solve it? (Hint: Think of solutions of the form $a_n = r^n$)

4. Solve the recurrence $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 0$ and $a_1 = 2$. Can you use the idea of $a_n = r^n$ from above? Why or why not? If not, how could you re-write the recurrence so that we can solve it?

2 Varsity Problems

1. Solve the recurrence $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ where $a_0 = 8$, $a_1 = 6$, and $a_2 = 26$.

2. How many ways are there to tile a 3×5 grid with 1×2 and 1×1 sized tiles (tiles can be rotated)?

3. (HMMT 2009) Simplify the product:

$$\prod_{m=1}^{100} \prod_{n=1}^{100} \frac{x^{n+m} + x^{n+m+2} + x^{2n+1} + x^{2m+1}}{x^{2n} + 2x^{n+m} + x^{2m}}.$$

   Express your answer in terms of x.

4. (AMC12 2001) Consider sequences of positive real numbers of the form $x, 2000, y, \ldots$ in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

5. The Fibonacci numbers are defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. Find a recurrence for the squared Fibonacci numbers F_n^2.

6. (AIME 2009) The sequence (a_n) satisfies $a_0 = 0$ and $a_{n+1} = \frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - a_n^2}$ for $n \geq 0$. Find the greatest integer less than or equal to a_{10}.

7. (Putnam 1996) Define a selfish set X to be such that $|X| \in X$ (i.e. has its size as an element). How many subsets of the set $\{1, \ldots, n\}$ are minimal selfish (with no proper selfish subset)?

8. (Putnam 1999) Let the sequence a_n be defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}, \qquad n \geq 4$$

Show that for all n, a_n is a multiple of n.

3 Varsity Challenge Problems

1. (Putnam 2002) The sequence a_n is defined by $a_0 = 1$, $a_{2n} = a_n + a_{n-1}$, $a_{2n+1} = a_n$. Prove that for any positive rational k there exists some m such that $k = \frac{a_m}{a_{m+1}}$.

2. (Putnam 2007) Let $x_0 = 1$ and for $n \geq 0$ let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}.

4 JV Warm-up

1. Suppose sequence $a_1, a_2, \ldots$ is defined as follows: $a_1 = 1$, $a_n = \frac{a_{n-1}}{a_{n-1}+1}$. Compute a_{2018}.

2. Evaluate the sum
$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{98 \cdot 99 \cdot 100}$$

3. Evaluate the sum
$$1 \cdot 4 + 4 \cdot 7 + 7 \cdot 11 + \cdots + 37 \cdot 40$$

5 JV Problems

1. Suppose $a_1, \ldots, a_{100}$ is a sequence with $a_1 = 0$ and $a_{n+1} - a_n = 7$ for all n. Compute $\sum_{n=1}^{100} a_n$.

2. How many positive, increasing, integer arithmetic progressions are there whose first five terms add up to 100?

3. A rocket is launched with initial speed of 100 mph. After every hour, it loses two thirds of its weight, and thus, its speed increases by a factor of 3. How many miles does the rocket travel in 6 hours?

4. Is there a geometric progression of natural numbers $t_1, t_2, \ldots, t_n$ such that
$$100 \leq t_1 < t_2 < \cdots < t_n \leq 1000$$
where $n = 5$? (Challenge: What about $n = 6$?)

5. Suppose the first 2018 terms of an arithmetic progression sum up to 1776 and first 1776 terms sum up to 2018. What is the sum of first $1776 + 2018 = 3794$ terms?

6. What is the sum of the following 42 terms in Arithmetic-Geometric progression?

$$4 + (4+3)\cdot 2 + (4+2\cdot 3)\cdot 2^2 + \cdots + (4+41\cdot 3)2^{41}$$

7. Compute $\displaystyle\sum_{r=1}^{20} r \cdot r!$

8. Compute $\displaystyle\sum_{r=1}^{18} \frac{1}{(r+2)r!}$

9. Given the recurrence $a_n = \frac{1}{4a_{n-1}-3}$ and $a_1 = 5$, compute a_{100}.

10. (AMC 2018) Suppose $a_1, a_2, \ldots, a_{2018}$ are numbers such that $a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}$. What is the remainder when $a_1^3 + a_2^3 + \ldots + a_{2018}^3$ is divided by 6?

6 JV Challenge Problems

1. (IMO 2014) Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that their exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}$$