Algebra | Misha Lavrov

Polynomials in a different basis

Western PA ARML Practice | *November 30, 2014*

Warm-up

1. Express $\binom{x}{3} = \frac{x(x-1)(x-2)}{3!}$ in the basis $\{1, x, x^2, x^3\}$. That is, find a_0, a_1, a_2, a_3 such that
$$\binom{x}{3} = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3.$$
2. Express x^3 in the basis $\{1, \binom{x}{1}, \binom{x}{2}, \binom{x}{3}\}$. That is, find a_0, a_1, a_2, a_3 such that
$$x^3 = a_0 \cdot 1 + a_1 \cdot \binom{x}{1} + a_2 \cdot \binom{x}{2} + a_3 \cdot \binom{x}{3}.$$
3. Express x^2 in the basis $\{(x-2)(x-4), (x-2)(x-7), (x-4)(x-7)\}$.
4. Express $\cos(4\theta)$ in the basis $\{1, \cos\theta, \cos^2\theta, \cos^3\theta, \cos^4\theta\}$.

Binomial coefficients

1. Find a formula for the *pentagonal numbers*, which begin $0, 1, 5, 12, 22, 35, 51, 70, \dots$.
2. A polynomial $P(x)$ of degree 2014 satisfies $P(k) = 2^k$ for $k \in \{0, 1, \dots, 2014\}$. Find $P(2015)$.
3. What if we replace 2 by an arbitrary value r in the above problem?
4. The formula for $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots + n^3$ is some degree 4 polynomial in n. Find the formula.
5. Using the identity $\displaystyle\sum_{n=0}^{\infty} \binom{n}{k} 2^{-n} = 2$, which holds for all values of k, find $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{2^n}$.
6. Prove the identity $\displaystyle\sum_{n=0}^{\infty} \binom{n}{k} 2^{-n} = 2$.

Interpolation

1. (Lagrange Interpolation) Let $x_0, \dots, x_n$ be distinct values. Define
$$\ell_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$
Then the unique polynomial of degree $\leq n$ that passes through the points (x_0, y_0), (x_1, y_1), $\dots$, (x_n, y_n) is the sum $y_0\ell_0(x) + y_1\ell_1(x) + \cdots + y_n\ell_n(x)$.

2. Find a quadratic polynomial $P(x)$ such that $P(2) = 11$, $P(4) = 29$, and $P(7) = 71$.
3. Find a general formula for all *cubic* polynomials $P(x)$ such that $P(2) = 11$, $P(4) = 29$, and $P(7) = 71$.
4. Simplify
$$\frac{(x-2)(x-4)}{(7-2)(7-4)} + \frac{(x-2)(x-7)}{(4-2)(4-7)} + \frac{(x-4)(4-7)}{(2-4)(2-7)}.$$
5. If $P(x)$ is a degree 4 polynomial that satisfies $P(k) = k^5$ for $k \in \{1, 2, 3, 5, 8\}$, find $P(0)$.
6. (USAMO 1975) A polynomial $P(x)$ of degree n satisfies $P(k) = \frac{k}{k+1}$ for $k \in \{0, 1, \ldots, n\}$. Find $P(n+1)$.

Miscellaneous

Horner's method for evaluating polynomials is to express $a_n x_n + a_{n-1}x^{n-1} + \cdots + a_0$ as
$$((\cdots((a_n \cdot x + a_{n-1}) \cdot x + a_{n-2}) \cdot x + a_{n-3}) \cdots + a_1) \cdot x + a_0$$
For example, $4x^3 + 3x^2 + 2x + 1$ becomes $((4 \cdot x + 3) \cdot x + 2) \cdot x + 1$.

1. A sequence (x_n) is defined by $x_0 = 1$ and
$$x_n = 2x_{n-1} + \binom{100}{n}.$$
Compute x_{100}.
2. (ARML 1995) Compute the largest prime factor of:
$$3 \cdot (3 \cdot (3 \cdot (3 \cdot (3 \cdot (3 \cdot (3 \cdot (3 \cdot (3 \cdot (3 \cdot (3+1)+1)+1)+1)+1)+1)+1)+1)+1)+1)+1.$$
3. What is the smallest possible degree of a polynomial that satisfies
$$P(1) = P(2),\ P(3) = P(4),\ P(5) = P(6),\ \ldots,\ P(2013) = P(2014)?$$
4. (Putnam 2005) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a.