Trigonometric Functions (Varsity)
Logarithmic Functions (JV)

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1 JV Warm-Up

The *logarithm* is an extremely useful function that is found all over math. For any positive b (except $b = 1$) we can define the function $\log_b(x)$ to be the unique number c such that $b^c = x$. (We often write $\log_b x$ and leave out the parentheses). A few examples:

- $\log_2 8 = 3$ since $2^3 = 8$.
- $\log_{10} 1,000,000 = 6$ since $10^6 = 1,000,000$.
- $\log_8 4 = \frac{2}{3}$ since $8^{2/3} = 4$.
- $\log_3 \frac{1}{9} = -2$ since $3^{-2} = \frac{1}{9}$.

You now know how to compute $\log_b(x)$ for many values of b and x. There are three important identities to know about logarithms:

1. $\log_b(x^a) = a \log_b(x)$.
2. $\log_b(xy) = \log_b(x) + \log_b(y)$.
3. $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ for any nonnegative $a \neq 1$. This is called the "change of base" formula.

Important warning. If a problem involves a log that you can't compute by hand, say $\log_2 3$, then don't try to compute it! Try to find a way to make it cancel out.

1.1 Warm-up Problems

1. (NYCIML F07) Compute $\log_2 3^2 \cdot \log_3 4^3 \cdot \log_4 5^4 \cdot \log_5 6^5 \cdot \log_6 7^7 \cdot \log_7 8^7$.

2. Derive identities (1) and (2) from the familiar facts of exponents: $(b^a)^c = b^{ac}$ and $b^{a+c} = b^a b^c$.

3. (NYCIML S06) You are given positive numbers r and s such that:

$$\log_4 r = \log_6 s = \log_9(2r + 3s).$$

Compute $\frac{r}{s}$.

2 JV Problems

1. Prove that $\log_b a = \frac{1}{\log_a b}$. This is a very useful identity for math competition problems. Use it in problems below!
2. Find a formula for $\log_{b^n} a$ in terms of n and $\log_b a$.
3. (AIME II 2013) Positive integers a and b satisfy
$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$
Compute the sum of all possible values of $a + b$.
4. (NYCIML F11) If $\log_{4n} 96 = \log_{5n} 75\sqrt{5}$, compute n^5.
5. (NYCIML ??) Compute the minimum possible value of $\log_y x + \log_{xy} y$ where $y > x \geq 1$.
6. (AMC 12A 2015) Compute the value of a such that
$$\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1.$$
7. (AIME II 2006) The lengths of the sides of a triangle with positive area are $\log_{10} 12$, $\log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n.
8. (Math Prize 2011) If $n > 10$, compute the greatest possible value of
$$\log n^{\log(\log(\log n))} - \log(\log n))^{\log(\log n)}$$
All the logarithms are base 10.
9. (NYCIML 2003) Compute all ordered triples (x, y, z) satisfying:
$$\log_2 x \log_2 y + \log_2(xy) = 2$$
$$\log_2 y \log_2 z + \log_2(yz) = 59$$
$$\log_2 z \log_2 x + \log_2(zx) = 4$$

3 JV Challenge Problems

Borrowed from Varsity practice. (Not about logarithms!)

1. (AIME II 2014) Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A) + \cos(3B) + \cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.
2. Let $x = \frac{180}{7}$ degrees. Find $\tan x \tan 2x \tan 3x$.

4 Varsity Warm-Up

Recall the basic trigonometric identities, namely

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \tag{1}$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{2}$$
$$\sin^2 x + \cos^2 x = 1 \tag{3}$$

and observe that we can recover most identities from these three via algebraic manipulation, using properties such as $\sin(-x) = -\sin x$ and $\frac{\sin x}{\cos x} = \tan x$. To solve trigonometric equations, it is often useful to apply these identities repeatedly (sometimes more cleverly than others) to reduce the equation into a solvable form.

The sine and cosine functions are also notable for their algebraic properties; namely, they are periodic with period 2π and are bounded in magnitude by 1. What does this mean? It means that $\sin\theta = \sin(2\pi + \theta)$ for all θ (verify this with (1)), and similarly for cosine. We also have that $|\sin\theta| \leq 1$ for all θ, with equality when θ is an odd integral multiple of $\frac{\pi}{2}$, and $|\cos\theta| \leq 1$ for all θ as well (when does equality hold)?

5 Varsity Warm-Up Problems

1. (AMC12B 2010) In $\triangle ABC$, $\cos(2A - B) + \sin(A + B) = 2$ and $AB = 4$. What is BC?
2. (AMC 12B 2014) What is the sum of all positive real solutions x to the equation
$$2\cos(2x)\left(\cos(2x) - \cos\left(\frac{2014\pi^2}{x}\right)\right) = \cos(4x) - 1?$$
3. (AIME II 2012) Let x and y be real numbers such that $\frac{\sin x}{\sin y} = 3$ and $\frac{\cos x}{\cos y} = \frac{1}{2}$. What is the value of $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$?

6 Varsity Problems

1. (AMC 12A 2004) If $\sum_{n=0}^{\infty} \cos^{2n}\theta = 5$, what is the value of $\cos 2\theta$?
2. (AMC 12P 2002) Let $f_n(x) = \sin^n x + \cos^n x$. For how many x in $[0, \pi]$ is it true that $6f_4(x) - 4f_6(x) = 2f_2(x)$?
3. (AMC 12B 2002) Let a and b be real numbers such that $\sin a + \sin b = \dfrac{\sqrt{2}}{2}$ and $\cos a + \cos b = \dfrac{\sqrt{6}}{2}$. Find $\sin(a+b)$.
4. Show that in a triangle $\triangle ABC$, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
5. (AIME I 2008) Find the positive integer n such that
$$\arctan\frac{1}{3} + \arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{n} = \frac{\pi}{4}.$$

6. (AIME II 2014) Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A)+\cos(3B)+\cos(3C)=1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

7. (AMC12A 2009) The first two terms of a sequence are $a_1=1$ and $a_2=\frac{1}{\sqrt{3}}$. For $n\geq 1$,

$$a_{n+2}=\frac{a_n+a_{n+1}}{1-a_na_{n+1}}.$$

What is $|a_{2009}|$?

8. Let $x=\frac{180}{7}$ degrees. Find $\tan x\tan 2x\tan 3x$.

9. (AIME I 2015) With all angles measured in degrees, the product $\prod_{k=1}^{45}\csc^2(2k-1)^\circ=m^n$, where m and n are integers greater than 1. Find $m+n$.

7 Varsity Challenge Problems

1. (CMIMC 2018) Compute

$$\sum_{k=0}^{2017}\frac{5+\cos\left(\frac{\pi k}{1009}\right)}{26+10\cos\left(\frac{\pi k}{1009}\right)}.$$

2. (AIME 2000) Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ\sin 46^\circ}+\frac{1}{\sin 47^\circ\sin 48^\circ}+\cdots+\frac{1}{\sin 133^\circ\sin 134^\circ}=\frac{1}{\sin n^\circ}.$$

3. (AIME I 2013) For $\pi\leq\theta<2\pi$, let

$$P=\frac{1}{2}\cos\theta-\frac{1}{4}\sin 2\theta-\frac{1}{8}\cos 3\theta+\frac{1}{16}\sin 4\theta+\frac{1}{32}\cos 5\theta-\frac{1}{64}\sin 6\theta-\frac{1}{128}\cos 7\theta+\dots$$

and

$$Q=1-\frac{1}{2}\sin\theta-\frac{1}{4}\cos 2\theta+\frac{1}{8}\sin 3\theta+\frac{1}{16}\cos 4\theta-\frac{1}{32}\sin 5\theta-\frac{1}{64}\cos 6\theta+\frac{1}{128}\sin 7\theta+\dots$$

so that $\frac{P}{Q}=\frac{2\sqrt{2}}{7}$. Then $\sin\theta=-\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

4. (HMMT 2014) Compute

$$\sum_{k=1}^{1007}\left(\cos\left(\frac{\pi k}{1007}\right)\right)^{2014}.$$