Sequences and Series — Misha Lavrov

Summing Series

Western PA ARML Practice — *December 4, 2016*

1 Useful identities

The following basic identities are ones you should learn without having to derive them every time you use them.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{1}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{2}$$

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n \tag{3}$$

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} \tag{4}$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \text{when } |x| < 1 \tag{5}$$

A more general form of (1) is that **the sum of a finite arithmetic series is equal to the number of terms multiplied by the average of the first and last term.**

You can often apply these identities more generally by factoring out a common term, or splitting a sum into two. For example, to find $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$, factor out 3 and then apply (5) with $r = \frac{1}{2}$.

The following are more advanced identities that still come up sometimes if you want more to learn.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{7}$$

$$\sum_{k=1}^{n} \binom{k}{r} = \binom{n+1}{r+1} \tag{8}$$

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(x-1)^2} \qquad \text{when } |x| < 1 \tag{9}$$

$$\sum_{k=0}^{\infty} \binom{k+r}{r} x^k = \frac{1}{(1-x)^{r+1}} \qquad \text{when } |x| < 1 \tag{10}$$

2 Switching the order of summation

1. Prove useful identity (9). (Hint: write kx^k as $\sum_{j=1}^{k} x^k$.)

2. (Concrete Mathematics[1]) Riemann's zeta function $\zeta(k)$ is defined to be the infinite sum

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots = \sum_{j=1}^{\infty} \frac{1}{j^k}.$$

   Find $\sum_{k=2}^{\infty}(\zeta(k) - 1)$.

3. We define the n^{th} *harmonic number* H_n to be the value of the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which has no closed form.

   Express $\sum_{k=1}^{n} H_k$ in terms of H_n.

4. (Concrete Mathematics) Find (again, in terms of H_n)

$$\sum_{k=1}^{n} \frac{H_k}{(k+1)(k+2)}.$$

5. Prove useful identity (6) by writing k^2 as $\sum_{j=1}^{k} k$. (Hint: some things will go wrong, but you can still save the day.)

6. Find $\sum_{k=1}^{n} k \cdot F_k$, where F_n is the n^{th} Fibonacci number: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. (Hint: solve problem #2 in the next section first.)

7. (ARML 1978) Find the sum of the infinite series $\sum_{k=1}^{\infty} \frac{k^2}{3^k}$.

8. (Putnam 2003) Show that for each positive integer n,

$$n! = \prod_{j=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/j \rfloor\}.$$

[1] Graham, Ronald L., Donald E. Knuth, and Oren Patashnik. "Concrete Mathematics: A Foundation for Computer Science." (1994). Chapter 2: Sums.

3 The method of differences

1. Find the sum $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{98\cdot 99} + \frac{1}{99\cdot 100}$.

2. Find $\displaystyle\sum_{k=1}^{n} F_k$, where F_n is the n^{th} Fibonacci number: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

3. (Wikipedia) A well-known (but hard) result is that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Find an approximation for this sum by using the upper bound
$$\sum_{k=1}^{\infty} \frac{1}{k^2} \le 1 + \sum_{k=2}^{\infty} \frac{1}{k^2 - 1/4}$$
and evaluating the sum on the right-hand side. (Bonus: what approximation for π do you get in this way?)

4. (ARML 1991) Let $\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots\left(1 - \frac{1}{1991^2}\right) = \frac{x}{1991}$. Compute the integer x.

5. (a) Prove useful identity (8).
   (b) We have $k^2 = 2\binom{k}{2} + \binom{k}{1}$. Use this, and useful identity (8), to derive useful identity (6).
   (c) Find a similar expression for k^3, and use it with useful identity (8) to derive useful identity (7). (Note: this method applies more generally to find the sum of *any* polynomial expression in k.

6. (a) Write the differences $\sin(n+1) - \sin n$ and $\cos(n+1) - \cos n$ in terms of $\sin n$, $\cos n$, and constants.
   (b) Find a function $f(n)$ such that $f(n+1) - f(n) = \sin n$.
   (c) Find a formula for $\displaystyle\sum_{k=1}^{n} \sin k$.

7. (VTRMC 2014) Find $\displaystyle\sum_{k=2}^{\infty} \frac{k^2 - 2k - 4}{k^4 + 4k^2 + 16}$.

8. (USAMO 1991) For any set S, let $\sigma(S)$ and $\pi(S)$ denote the sum and product, respectively, of the elements of S, with $\sigma(\emptyset) = 0$ and $\pi(\emptyset) = 1$. Prove that
$$\sum_{S \subseteq [n]} \frac{\sigma(S)}{\pi(S)} = (n^2 + 2n) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)(n+1),$$
where the sum ranges over all subsets S of $[n] = \{1, 2, 3, \ldots, n\}$.

9. (a) Find $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{2^{2^k} + 1}$.
   (b) Show that $\displaystyle\sum_{\text{all } k \ge 1} \frac{k}{2^k + 1} = \sum_{\text{odd } k \ge 1} \frac{k}{2^k - 1}$.