Functional Equations

Advanced Topics Practice 10/6/19

Anish Sevekari

1 Problem Set

1. Let $f : \mathbb{Q} \to \mathbb{Q}$ be such that $f(0) = 0$ and $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$. Find all possible values of f. What changed if we remove the condition $f(0) = 0$? (Food for thought: Does same result hold if $f : \mathbb{R} \to \mathbb{R}$ instead?)

2. Find all $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) = f(x) + f(y)$. What if $f : \mathbb{R}^* \to \mathbb{R}$?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{Q}$. Find all possible value of f.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) \leq f(x) + f(y)$ and $f(x) \leq x$.

5. (INMO 2011) Let $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + y)f(x - y) = (f(x) + f(y))^2 - 4x^2 f(y)$$

6. (INMO 2012) Let $f : \mathbb{Z} \to \mathbb{Z}$ be function satisfying $f(0) \neq 0$, $f(1) = 0$ and

   (i) $f(xy) + f(x)f(y) = f(x) + f(y)$
   (ii) $(f(x - y) - f(0))f(x)f(y) = 0$

   for all $x, y \in \mathbb{Z}$.

   (a) Find the set of all possible values of f.
   (b) If $f(10) \neq 0$ and $f(2) = 0$, find the set of all integers n such that $f(n) \neq 0$.

7. (IMOSL 2018 A1)Let $\mathbb{Q}_{>0}$ denote the set of rational numbers strictly bigger than zero. Determine all the functions satisfying

$$f(x^2 f(y)^2) = f(x^2)f(y)$$

   for all $x, y \in \mathbb{Q}_{>0}$.

8. (IMOSL 2018 A5)Find all $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

9. (IMOSL 2017 A6)Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any reals x, y

$$f(f(x)f(y)) + f(x + y) = f(xy)$$