Logarithms

JV Practice 10/6/19

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1 Useful Formulas

- Definition: If $a^x = b$, then $x = \log_a(b)$.
- Convention: log is base 10, ln is base e, though sometimes this is broken.
- Multiplication: $\log_a(xy) = \log_a(x) + \log_a(y)$
- Division:[1] $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- Exponentiation: $\log_a(x^y) = y \log_a(x)$
- Change of Base: $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ (key to remembering: the <u>base</u> goes on the <u>bottom</u>)
- Fun identity (follows from Change of Base): $\log_a(b) = \frac{1}{\log_b(a)}$

2 Warmup

1. (2003 AMC 12B Problem 17) If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?
2. (2005 AMC 10B Problem 17) Suppose that $4^a = 5$, $5^b = 6$, $6^c = 7$, and $7^d = 8$. What is $a \cdot b \cdot c \cdot d$?
3. (2010 AMC 12A Problem 11) The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

3 Problems

1. (NYCIML F10B25) Compute $(\log_{125} 16)(\log_4 27)(\log_3 625)$.
2. (NYCIML F06B07) Compute
$$\frac{\log(8)}{\log \frac{1}{8}}$$
3. (NYCIML S11B26) Let $\log_{10} 70 = m$ and $\log_{10} 20 = p$. Given that $\log_{10} 14 = Am + Bp + C$ where A, B, and C are integers, compute the ordered triple (A, B, C).
4. (NYCIML F06A19) If $\log_b(a)\log_c(a)\log_c(b) = 25$ and $\frac{a^2}{c^2} = c^k$, what is the sum of all possible values of k?

[1] Technically this follows from multiplication and exponentiation

5. (2018 AMC 12B Problem 7) What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

6. (2002 AMC 12B Problem 22) For all integers n greater than 1, define $a_n = \frac{1}{\log_n 2002}$. Let $b = a_2 + a_3 + a_4 + a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. What is $b - c$?

7. (2008 AMC 12A Problem 16) The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12^{th} term of the sequence is $\log b^n$. What is n?

8. (2019 AMC 12A Problem 15) Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$$

and all four terms on the left are positive integers, where log denotes the base 10 logarithm. What is ab?

9. (1984 AIME Problem 5) Determine the value of ab given

$$\log_8 a + \log_4 b^2 = 5$$
$$\log_8 b + \log_4 a^2 = 7$$

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10. (David Altizio, Mock AMC 10/12 2013) Suppose x and y are real numbers such that $\log_x(y) = 6$ and $\log_{2x}(2y) = 5$. What is $\log_{4x}(4y)$?

11. (2000 AIME II Problem 1) The number $\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

4 Challenge Problems

1. The domain of the function $f(x) = \log_{\frac{1}{2}}(\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}} x))))$ is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

2. Let S be the set of ordered triples (x, y, z) of real numbers for which

$$\log_{10}(x + y) = z \text{ and } \log_{10}(x^2 + y^2) = z + 1.$$

There are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. What is the value of $a + b$?

3. The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?

4. Let $m > 1$ and $n > 1$ be integers. Suppose that the product of the solutions for x of the equation

$$8(\log_n x)(\log_m x) - 7 \log_n x - 6 \log_m x - 2013 = 0$$

is the smallest possible integer. What is $m + n$?

5. (David Altizio, Mock AIME I 2015) Suppose that x and y are real numbers such that $\log_x(3y) = \frac{20}{13}$ and $\log_{3x}(y) = \frac{2}{3}$. The value of $\log_{3x}(3y)$ can be expressed in the form $\frac{a}{b}$ where a and b are positive relatively prime integers. Find $a + b$.