

Roots of Unity

Varsity Practice 9/29/19

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1 Warmup

1. **DeMoivre's Theorem:** $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ for all integers n .
2. **Euler's Formula:** $e^{i\theta} = \cos(\theta) + i \sin(\theta)$. This lends itself to $e^{i\pi} = -1$.
3. **Roots of Unity:** Define $\zeta_{k,n} = e^{i2\pi \frac{k}{n}}$. Note that $\zeta_{k,n}^n = 1$ for all integer k .
4. Compute $\sum_{k=1}^{50} \sin(k\pi/25)$.
5. Suppose a satisfies $x^{15} - 1 = 0$ and b satisfies $x^{20} - 1 = 0$. Find the smallest $n > 0$ such that $(ab)^n - 1 = 0$ must be true.
6. Show that $\sum_{k=0}^n \zeta_{k,n} = 0$. In addition, show that $\sum_{k=0}^n \zeta_{k,n}^j = 0$ for j not a multiple of n , and n otherwise.

2 Problems

1. (CHMMC 2017 Team) Let $a = e^{4\pi i/5}$ be a nonreal fifth root of unity and $b = e^{2\pi i/17}$ be a nonreal seventeenth root of unity. Compute the value of the product

$$(a+b)(a+b^{16})(a^2+b^2)(a^2+b^{15})(a^3+b^8)(a^3+b^9)(a^4+b^4)(a^4+b^{13})$$

2. (BMT 2015 Ana) Let ω be a primitive 7th root of unity. Find $\prod_{k=0}^6 (1 + \omega^k - \omega^{2k})$.
3. (HMMT 2008 Alg) Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a, b .
4. (HMMT 2018 Alg) Let α, β, γ be three real numbers. Suppose that

$$\cos \alpha + \cos \beta + \cos \gamma = 1$$

$$\sin \alpha + \sin \beta + \sin \gamma = 1$$

Find the smallest possible value of $\cos \alpha$

5. (HMMT 2017 Guts) Let a and b be complex numbers satisfying the two equations

$$a^3 - 3ab^2 = 36$$

$$b^3 - 3ba^2 = 28i$$

Let M be the maximum possible magnitude of a . Find all a such that $|a| = M$.

6. (HMMT 2018 Guts) Michael picks a random subset of the complex numbers $\{1, \omega, \omega^2, \dots, \omega^{2017}\}$ where ω is a primitive 2018th root of unity and all subsets are equally likely to be chosen. If the sum of the elements in his subset is S , what is the expected value of $|S|^2$? (The sum of the elements of the empty set is 0.)
7. (BMT 2013 Ana) The minimal polynomial of a complex number r is the unique polynomial with rational coefficients of minimal degree with leading coefficient 1 that has r as a root. If f is the minimal polynomial of $\cos(\pi/7)$, what is $f(-1)$?
8. (HMMT 2014 Guts) Compute $\sum_{k=0}^{1007} (\cos(\frac{\pi k}{1007}))^{2014}$
9. (BMT 2016 Individual) Find $\prod_{k=1}^{2017} e^{\pi i k / 2017} 2 \cos(\frac{\pi k}{2017})$
10. (SMT 2013 Alg) Given z a complex number such that $z^{13} = 1$, find all possible values of $z + z^3 + z^4 + z^9 + z^{10} + z^{12}$.
11. (HMMT 2013 Alg) Let z be a non-real complex number such that $z^{23} = 1$. Evaluate $\sum_{k=0}^{22} \frac{1}{1+z^k+z^{2k}}$.
12. (HMIC 2014) Let ω be a root of unity and f be a polynomial with integer coefficients. Show that if $|f(\omega)| = 1$, then $f(\omega)$ is also a root of unity.