Algebra Review

Varsity Practice 10/13/19

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1

Let $P(x)$, $Q(x)$ be nonconstant polynomials with real number coefficients. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor$$

for all real numbers y, then $P(x) = Q(x)$ for all real numbers x.

2

Suppose a, b, c, and d are positive real numbers that satisfy the system of equations

$$(a+b)(c+d) = 143,$$
$$(a+c)(b+d) = 150,$$
$$(a+d)(b+c) = 169.$$

Compute the smallest possible value of $a^2 + b^2 + c^2 + d^2$.

3

Let Q be a polynomial

$$Q(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where $a_0, \ldots, a_n$ are nonnegative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

4

If a, b, c represent the lengths of the sides of a triangle, prove the inequality:

$$3 \le \sqrt{\frac{a}{-a+b+c}} + \sqrt{\frac{b}{-b+a+c}} + \sqrt{\frac{c}{-c+b+a}}.$$

5

Find the $x \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$x^2 + x \in \mathbb{Z} \text{ and } x^3 + 2x^2 \in \mathbb{Z}$$

6

Find the complex numbers z_1, z_2, z_3 of same absolute value having the property that:

$$1 = z_1 z_2 z_3 = z_1 + z_2 + z_3.$$

7

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

8

Let x, y and z be positive real numbers such that $xy + yz + xz = 3xyz$. Prove that

$$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$

and determine when equality holds.

9

For any $n \in \mathbb{N}$, define the $n-$th cyclotomic polynomial to be

$$\Phi_n = \prod_{1 \leq k \leq n, \gcd(k,n)=1} (x - \xi_k),$$

where ξ_k are the $n - th$ roots of unity $\xi_k = e^{2\pi ik/n}$. In particular, ξ_k are $n-$th roots of unity, but not $m-$th roots of unity, $m < n$, since $\gcd(k, n) = 1$. Prove that:

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

and if $x > 1, n \in \mathbb{N}$

$$(x - 1)^{\phi(n)} \leq \Phi_n(x) \leq (x + 1)^{\phi(n)},$$

where $\phi(n)$ is the number of positive integers less than n relatively prime to it (Euler's totient function).