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Optimization Problems

Western PA ARML Practice *April 9, 2017*

1 Derivatives

1.1 Properties

Some useful derivative formulas to know (a is a constant, all angles are radians):

$$f(x) = x^a, a \neq 0 \qquad f'(x) = a \cdot x^{a-1} \tag{1}$$
$$g(x) = a \qquad g'(x) = 0 \tag{2}$$
$$h(x) = a^x \qquad h'(x) = a^x \ln a \tag{3}$$
$$j(x) = \sin x \qquad j'(x) = \cos x \tag{4}$$
$$k(x) = \cos x \qquad k'(x) = -\sin x \tag{5}$$
$$\ell(x) = \ln x \qquad \ell'(x) = \frac{1}{x} \tag{6}$$

And some useful rules for combining them:

$$g(x) = a \cdot f(x) \qquad g'(x) = a \cdot f'(x) \tag{7}$$
$$h(x) = f(x) + g(x) \qquad h'(x) = f'(x) + g'(x) \tag{8}$$
$$h(x) = f(x) \cdot g(x) \qquad h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \tag{9}$$
$$h(x) = f(g(x)) \qquad h'(x) = f'(g(x)) \cdot g'(x) \tag{10}$$
$$h(x) = \frac{f(x)}{g(x)} \qquad h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2} \tag{11}$$
$$g(x) = \prod_{i=1}^{n} f_i(x) \qquad g'(x) = g(x) \cdot \sum_{i=1}^{n} \frac{f_i'(x)}{f_i(x)} \tag{12}$$

1.2 Exercises

1. Take the derivative.
   (a) If $f(x) = \tan x$, then $f'(x) =$
   (b) If $f(x) = e^{e^{e^x}}$, then $f'(x) =$
   (c) If $f(x) = \frac{\ln x}{\ln \ln x}$, then $f'(x) =$
   (d) If $f(x) = x^x$, then $f'(x) =$

2. Knowing that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all x with $|x| < 1$, find a formula for $\sum_{n=1}^{\infty} nx^n$, and use it to evaluate the sum $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

3. The n^{th} derivative $f^{(n)}(x)$ is just what you get by taking the derivative of $f(x)$, then taking the derivative of that, and so on, n times. If $f(x) = e^x \sin x$, find $f^{(100)}(x)$, the 100^{th} derivative of $f(x)$.

4. Find a function $f(x)$ such that $f'(x) = \ln x$.

2 Optimization

2.1 Setting derivatives to 0

Often, we want to find the value of x for which $f(x)$ is as large or as small as possible. The key to doing this with calculus is the following:

- If $f'(x) > 0$ for all x such that $a \le x \le b$, then $f(x)$ is increasing between a and b: $f(a) < f(x) < f(b)$ for all x such that $a < x < b$.
- If $f'(x) < 0$ for all x such that $a \le x \le b$, then $f(x)$ is decreasing between a and b: $f(a) > f(x) > f(b)$ for all x such that $a < x < b$.
- Let x^* be the point such that $f(x^*)$ is the largest value of $f(x)$ for all x such that $a \le x \le b$. Then either $x^* = a$ or $x^* = b$ or $f'(x^*) = 0$.

Note: the converse of the last rule does not hold. For example, if $f(x) = x^3$, then $f'(0) = 0$, but 0 is neither a minimum nor a maximum of $f(x)$.

2.2 Exercises

1. Which of the values

$$1, 2^{1/2}, 3^{1/3}, 4^{1/4}, 5^{1/5}, 6^{1/6}, 7^{1/7}$$

is the largest? Which is the smallest?

2. Some classic exercises:

   (a) You have 100 feet of fencing. If you want to fence off a rectangular region, what is the largest area you can enclose?

   (b) What if you can use one side of an infinitely large barn to close the region (so you don't have to fence off that side)?

   (c) Starting from a 10×16 sheet of cardboard, four $x \times x$ squares are cut out from the corners; then the cardboard is folded to make a box (with no top). What is the largest possible volume of the box?

3. Find the point on the parabola $y = x^2$ closest to the point $(3, 6)$.

4. (ARML 1995) A trapezoid has a height of 10, its legs are integers, and the sum of the sines of the acute base angles is $\frac{1}{2}$. Compute the largest possible sum of the lengths of the two legs.

5. (a) Prove that $e^x \geq 1 + x$ for all x, with equality only at $x = 0$.

   (b) In fact, prove that for $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2}$, with equality only at $x = 0$.

   (c) In fact, prove that for $x \geq 0$ and for all integers $k \geq 0$,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!}.$$

6. (VTRMC 1991) Prove that if $x > 0$ and $n > 0$, where x is real and n is an integer, then

$$\frac{x^n}{(x+1)^{n+1}} \leq \frac{n^n}{(n+1)^{n+1}}.$$

7. A function $f(x)$ satisfies the property that $f'(x) = \frac{1}{x}$ for all x, and $f(1) = 0$. (This is one possible definition of the natural logarithm $\ln x$, but you should pretend you don't know that $f(x) = \ln x$ yet: don't use any other properties of $\ln x$.)

   Prove that $f(xy) = f(x) + f(y)$ for all x and y.

   (Hint: what can you say about the function $f(cx) - f(x)$, when c is a constant?)

8. (VTRMC 2013) Let $\triangle ABC$ be a right triangle with $\angle ABC = 90°$, and let D be a point on AB such that $AD = 2DB$. What is the maximum possible value of $\angle ACD$?

9. (a) Prove that for $0 < x < 1$, $\sin x > \frac{x}{1+x}$.

   (b) Use this to show that the sum

$$\sin 1 + \sin\sin 1 + \sin\sin\sin 1 + \sin\sin\sin\sin 1 + \cdots$$

   diverges.

   (c) By proving a better inequality, find the largest value of α for which

$$\frac{\sin 1}{1^\alpha} + \frac{\sin\sin 1}{2^\alpha} + \frac{\sin\sin\sin 1}{3^\alpha} + \frac{\sin\sin\sin\sin 1}{4^\alpha} + \cdots$$

   still diverges.