Combinatorics Victor Xu

Solutions to Problems

Western PA ARML Practice *November 22, 2015*

1 Solutions

Permutations

1. Five runners run a race. How many different ways can they finish?
   There are 5 people who could finish first, 4 who could finish second, and so on, for a total of $5! = 5 * 4 * 3 * 2 * 1 = \boxed{120}$ different ways to finish.

2. How many distinct ways are there to rearrange the letters in REARRANGE?
   There are 9 letters in REARRANGE, so there are 9! ways to rearrange these letters. However, there are 2 E's, 2 A's, and 3 R's that can cause duplicate arrangements, so we need to divide by $2! * 2! * 3!$. This gives us a final total of $\boxed{15120}$.

Binomial Coefficients

1. Five runners run a race. How many different ways can the top three finishers be selected, if we do not care about the specific order of these top three?
   Again, there are 5 people who could finish first, 4 who could finish second, and 3 who could finish third. However, there are $3! = 6$ ways to select any unique group of three using this method, so our final result should be $\frac{5*4*3}{6} = \boxed{10}$

Stars and Bars

1. Ten identical robots run a race with three different finish lines. How many different ways can the robots be distributed among the finishing lines?
   The robots are the stars, and we need two bars to differentiate which robots will go to which finish lines. This gives a total result of $\binom{12}{2} = \boxed{66}$

Problems

1. In order to list the numbers 0 to 7 in binary notation, we need 12 1's (000, 001, 010, 011, 100, 101, 110, 111). How many 1's are needed to list in binary the numbers from 0 to 1023?
   We can match each number x to the number $1023 - x$. Each pair of numbers will have 10 1's, and there are 512 total pairs, for $\boxed{5120}$ total 1's.

2. Try to justify/prove to yourself the validity of the formulas given.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{1}$$

   One way to look at this equation is to see that $\frac{n!}{(n-r)!}$ is the number of ways of selecting r elements from n distinct elements where different orders of selecting the elements are treated as different arrangements, and then there are r! ways any set of r objects could be ordered, so we divide by r!.

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \tag{2}$$

Imagine $(x+y)^n$ as $(x+y)(x+y)(x+y)$... Now, to get a specific $x^i y^{n-i}$ out of this product, we must multiply i x's with n-i y's. There are $\binom{n}{i}$ ways to choose i of these sums that will contribute an x, so the coefficient of $x^i y^{n-i}$ is $\binom{n}{i}$.

3. There is a bag with 50 red balls, 50 blue balls, and 30 yellow balls. Given that after pulling out 65 balls at random there are 5 more red balls then blue balls, what is the probability that the next ball pulled out is red? Suppose we had some probability p of choosing 5 more red balls than blue balls if we chose y yellow balls. Then p is also the probability of choosing 30-y yellow balls and 5 more blue balls than red balls, since the probability of choosing any set of balls is the same as the probability of choosing the balls not in that set. Additionally, the probability of choosing y yellow balls, r red balls, and b blue balls is the same as the probability of choosing y yellow balls, b red balls, and r red balls since there are 50 blue balls and also 50 red balls. Combining these two facts, we find that given that we choose 5 more red balls than blue balls, the probability of choosing y yellow balls is the same as the probability of choosing 30 - y yellow balls. If we look at these pairs of cases as a group, each group will have an expected amount of 15 yellow balls already chosen, leaving an average of $\frac{45}{2}$ red balls left in the bag, for a final probability of $\frac{45}{2} * \frac{1}{65} = \boxed{\frac{9}{26}}$.