Induction

JV Practice 4/10/20

Da Qi Chen

1 Warm-Up Problems

1. Prove that the sum of the first n positive integers is $n(n+1)/2$.
2. Prove that the sum of the first n positive odd integers is n^2.
3. A pizza is cut into n pieces such that each cut is a straight line through the pizza (not necessarily through the center). Prove that one can always flip certain pieces face-down such that no two adjacent slices face the same orientation.

2 Problem Set

1. Prove that any number larger than 13 can be written as the sum of multiples of $5, 7$ and 8.
2. Let $a_1 = a_2 = 1, a_n = a_{n-1} + an - 2$ for $n \geq 3$ be the sequence of Fibonacci numbers. Prove that $a_1 + ...a_n = a_{n+2} - 1$.
3. Given the same Fibonacci sequence as above, prove that $a_n < (\frac{7}{4})^{n-1}$ for all $n \geq 2$.
4. Suppose in a tournament where every team plays against one another exactly once, no game ends in a tie. Prove that there exists a team W where everyone else either lost to W or lost to someone that lost to W.
5. A finite binary sequence is written on the board. Every turn, you can either flip the first digit, or flip the digit immediately after the first occurrence of 1. For example, 001011 can be turned into either 101011 or 001111. Prove that any sequence can be turned in finite steps into another sequence of the same length.
6. Prove that any positive integer can be uniquely written as the product of an odd number and a power of 2.
7. Prove that a $2^n \times 2^n$ chessboard with one square missing can always be perfectly tiled by L-shaped 3-piece tiles.
8. Given a rectangular grid-like chocolate bar, you can break it into two pieces by cutting along any of the horizontal/vertical grid lines. If the bar has dimension $m \times n$, find how many breaks you need to turn it into mn individual unit pieces. (Note once a chocolate is cut, you can no longer break both pieces simultaneously with one cut).
9. Prove that $a^{2^n} - 1$ is divisible by 2^{n+2} for all positive integer n and positive odd integer a.