Functional Equations

Varsity Practice 04/12/2020

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1 Warm-Up Problems

1. Consider the recurrence $a_{n+1} = 2a_n + 1$, with $a_0 = 0$.
   (a) Compute a_1, a_2, a_3, a_4, a_5. Can you guess the value of a_n?
   (b) Prove that $a_n = 2^n - 1$.

2. Consider the recurrence $a_{n+1} = 2a_n + n$, with $a_0 = 1$. Compute $a_1, a_2, \ldots, a_5$. Can you guess the formula for a_n?

3. Let A_n denote the number of ways of tiling a $1 \times n$ box using blocks of size 1×1 and 1×2.
   (a) Find A_1, A_2, A_3, A_4, A_5. Is the sequence familier to you?
   (b) Compute a recursive expression for A_n. That is, express A_n in terms of A_{n-1} and A_{n-2}.
   (c) (Bonus:) Can you find a closed form formula for A_n?

2 Problems

A generating function for the series $a_0, a_1, \ldots$ is a power series given by

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

Sometimes, recurrence relations between a_i can translate into algebraic identities for $A(x)$, allowing us to compute a closed form expression for $A(x) = \frac{P(x)}{Q(x)}$ where $P(X)$ and $Q(x)$ are polynomials. This can help us in solving for a_n.

Some useful identities:

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

$$\frac{dA(x)}{dx} = \sum_{i=0}^{\infty} a_{i+1}(i+1)x^i$$

the second identity says that you can differentiate each part individually.

(a) Find the generating function in simple form of
   (a) $a_n = 1$.
   (b) $a_n = n$.
   (c) $a_n = \alpha n + \beta$.

(b) Use generating functions to solve the recurrence $a_{n+1} = 2a_n + 1$.

(c) Consider the Fibonacci Sequence given by the recurrence $A_n = A_{n-1} + A_{n-2}$, with $A_0 = 1, A_1 = 1$.
   (a) Use the recurrence above to get an identity for $F(x) = \sum_{i=0}^{\infty} A_n x^n$.

   (b) Solve the identity to get a simple form expression for $F(x)$.
   (c) How can you use this expression to compute values of A_n?

(d) Catalan Numbers: Let C_n denote the number of paths from $(0,0)$ to (n,n) consisting of $2n$ steps, where each step is of 1 unit along positive x or y directions, which do not go above the diagonal (they can touch diagonal)
   (a) Find a recurrence relation for C_n.
   (b) Convert the recurrence relation into an identity for $A(x) = \sum_{i=0}^{\infty} C_n x^n$.
   (c) Find closed form for the generating function $A(x)$.
   (d) Find a closed form for C_n.