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The Twelvefold Way

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The Twelvefold Way

Suppose we consider the following class of problem: we want to count the number of functions $f : \{1, 2, 3, \ldots, X-1, X\} \to \{1, 2, 3, \ldots, Y-1, Y\}$. Depending on the specific instance, we might want to introduce one or more requirements:

- The functions f might have to be injective (no value in $\{1, \ldots, Y\}$ is obtained more than once) or surjective (every value in $\{1, \ldots, Y\}$ is obtained at least once).
- Order might not matter for $\{1, \ldots, Y\}$: it only matters which elements of $\{1, \ldots, X\}$ are sent to the same value.
- Order might not matter for $\{1, \ldots, X\}$: it only matters how many elements of $\{1, \ldots, X\}$ are sent to a given $y \in \{1, \ldots, Y\}$.

This yields twelve cases, as in the table below. I've given you twelve problems, each of which is an example of a different case. When you solve the problems, fill in the table with problem numbers.

Does order matter?	yes	for $\{1, \ldots, X\}$	for $\{1, \ldots, Y\}$	no
Any f		#2		
Injective f	#1			
Surjective f			#3	

Problems

1. After a competition in which 100 people participate, assuming there are no ties, in how many ways can 1st, 2nd, and 3rd place be assigned?
2. An *equivalence relation* on $\{1, \ldots, X\}$ is a kind of "generalized equality": for some elements $1 \le x_1, x_2 \le X$, we define $x_1 \equiv x_2$, however we like, but with the requirements that:
   - $x \equiv x$ for all x.
   - If $x_1 \equiv x_2$, then $x_2 \equiv x_1$.
   - If $x_1 \equiv x_2$ and $x_2 \equiv x_3$, then $x_1 \equiv x_3$.

   An equivalence relation groups elements of $\{1, \ldots, X\}$ into *equivalence classes*: groups of elements any two of which are equal. (For example, if $x_1 \equiv x_2$ means $x_1 - x_2$ is even, then $\{1, 2, 3, 4, 5\}$ has equivalence classes $\{1, 3, 5\}$ and $\{2, 4\}$.)

   In terms of X, how many equivalence relations on $\{1, \ldots, X\}$ have at most $Y = 2$ equivalence classes?

3. Given $X = 15$ gold pieces and $Y = 5$ pirates, find the number of ways to distribute the gold to the pirates with the requirement that each pirate must receive at least one gold piece.

4. Instead of gold pieces, suppose the $Y = 5$ pirates are dividing up $X = 8$ rare gems, each with its own unique and bloody history, so that the pirates care which gems they get. How many ways are there to distribute the gems so that each pirate receives a gem?

5. This is a silly case of the twelvefold way. I have X identical pigeons and Y identical holes, and I would like to put each pigeon into a different hole. How many distinct ways are there to do this, in terms of X and Y?

6. How many equivalence relations on $\{1, \ldots, 6\}$ have exactly 3 equivalence classes?

7. (Po-Shen Loh) A word over the alphabet $\{a, b, c, \ldots, z\}$ is called *increasing* if its letters appear in alphabetical order. For example, *boost* is increasing, but *hinder* is not. How many increasing "words" (sequences of letters) are there of length 52?

   The answer is large, so you should leave it unevaluated.

8. How many ways are there to write 16 as a sum of four positive integers $a + b + c + d$? Here, addition is commutative, so $3 + 5 + 3 + 5$ and $5 + 5 + 3 + 3$ count as the same sum.

9. This is another silly case of the twelvefold way. I have X pigeons and Y identical holes, but I have given names to all of my pigeons and can tell them apart. However, I would still like to put each pigeon into a different hole. How many distinct ways are there to do this, in terms of X and Y?

10. (ARML 1990) Compute the number of integers between 100000 and 1000000 with the property that their digits are distinct and increase from left to right.

11. (Putnam 1985) Find the number of ordered triples of sets (A, B, C) such that $A, B, C \subseteq \{1, \ldots, 10\}$ with
    (a) $A \cup B \cup C = \{1, \ldots, 10\}$, and
    (b) $A \cap B \cap C = \emptyset$.

12. Suppose I look at all the equivalence relations on the set $\{1, 2, \ldots, 10\}$ that have 3 or fewer equivalence classes. I define an equivalence relation $\cong$ on the set of these equivalence relations, where I declare two equivalence relations to be equivalent if some permutation of $\{1, 2, \ldots, 10\}$ transforms one into the other. For example, the equivalence relation

    $$x_1 \equiv x_2 \Leftrightarrow x_1 - x_2 \text{ is even} \qquad \text{i.e. } 1 \equiv 3 \equiv 5 \equiv 7 \equiv 9 \text{ and } 2 \equiv 4 \equiv 6 \equiv 8 \equiv 10$$

    is equivalent under $\cong$ to the equivalence relation

    $$x_1 \sim x_2 \Leftrightarrow \left\lceil \frac{x_1}{5} \right\rceil = \left\lceil \frac{x_2}{5} \right\rceil \qquad \text{i.e. } 1 \sim 2 \sim 3 \sim 4 \sim 5 \text{ and } 6 \sim 7 \sim 8 \sim 9 \sim 10$$

    because $\sim$ can be transformed into $\equiv$ by relabeling $1, 2, 3, 4, 5$ as $1, 3, 5, 7, 9$ and $6, 7, 8, 9, 10$ as $2, 4, 6, 8, 10$.

    (If we were feeling perverse, we might write this as $\equiv \cong \sim$.)

    How many equivalence classes does $\cong$ have?