Pigeonhole Principle

Tudor Popescu and Ariel Uy

1 JV

2 Warm-Up

1. Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

2. Given any 5 points in a square with side length 1 prove that there are 2 points whose distance is at most $\frac{1}{\sqrt{2}}$.

3 Problems

1. Without looking, spell the title.[1]

2. CMU's Scotty dog colors every point on the plane either red or black. Prove that no matter how the coloring is done, there must exist two points, exactly a mile apart, that are the same color.

3. Prove that for any set of five integers, there are three integers whose sum is divisible by 3.

4. (Putnam 1978) Let A be any set of exactly 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \cdots, 97, 100$. Prove there must be at least two distinct integers in A whose sum is 104.

5. (AIME 1989) When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Compute the probability of getting heads exactly three times.

6. 31 balls, some black and some white, are divided into five jars. Does one of the jars necessarily contain at least four balls of the same colour?

7. 2018 students attended today's ARML practice. Since all of them are great friends, they all hugged each other at the beginning of the practice. Show that at each moment there are at least 2 students who have hugged the same number of people.

8. Prove that if 51 integers are chosen from the set $\{1, 2, \ldots, 100\}$, one of them must be divisible by another.

9. Prove that if a 3×7 grid of points is colored red and blue, there will always be a rectangle whose 4 corners have the same color.

[1] Shamefully stolen from Po-Shen Loh's MOP handout

3.1 Homework

1. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, prove that there is a rational number $\frac{p}{q}$, with $1 \leq q \leq n$, such that $|x - \frac{p}{q}| < \frac{1}{qn}$.
2. The squares of an 8×8 checkerboard are filled with the numbers $\{1, 2, ..., 64\}$. Prove that there are two adjacent squares (sharing a side) containing numbers that differ by at least 5.

4 Varsity

5 Warm-Up

1. Given 7 points in a regular hexagon of length 1, show that there are 2 points whose distance is less than 1.
2. Prove that any set of 17 integers contains five integers whose sum is divisible by 5.

6 Problems

1. Let n be a positive integer and let S be a set containing n integers. Show that there is a nonempty subset T of S such that the sum of the elements of T is divisible by n.
2. Colour every point in the plane red or blue. Show that some equilateral triangle has all of its vertices the same color.
3. Let n be a positive integer that is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.
4. (AIME 2004) How many positive integer divisors of 2004^{2004} have exactly 2004 divisors?
5. Let set S contain five integers each greater than 1 and less than 120. Show that S contains a prime or two elements of S share a prime divisor.
6. (Paul Erdos) Prove that, if $n+1$ integers are chosen from the set $\{1, 2, ..., 2n\}$, one of them must be divisible by another.
7. (AIME 1999) Forty teams play a tournament in which every team plays every other team exactly once. No ties occur, and each team has a 50% chance of winning any game it plays. Compute the probability that no two teams win the same number of games.
8. Let $n > 1$ and a be integers such that $\gcd(a, n) = 1$. Then there are integers x, y satisfying $0<|x| \leq \sqrt{n}$ and $0 < y \leq \sqrt{n}$ such that $ay - x$ is divisible by n. *This is Tudor's favorite lemma, commonly known as Thue's lemma.*
9. Let $p > 5$ be a prime such that p divides $k^2 + 5$ for some integer k. Then there are positive integers x, y such that $p^2 = x^2 + 5y^2$. *Hint: use the previous problem.*
10. (Putnam 1993/A4). Let $x_1, \ldots, x_{19}$ be positive integers less than or equal to 93. Let $y_1, \ldots, y_{93}$ be positive integers less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_i's.

6.1 Homework

1. (Dirichlet approximation) For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, prove that there is a rational number $\frac{p}{q}$, with $1 \leq q \leq n$, such that $|x - \frac{p}{q}| < \frac{1}{qn}$.

2. (Erdos-Szekeres theorem) Let m, n be natural numbers. The numbers $1, 2, 3, ..., mn + 1$ are arranged in some order. Prove that there exists either a subsequence of $m + 1$ terms in increasing order, or a subsequence of $n+1$ terms in decreasing order. (Terms in a subsequence do not have to be consecutive.)