Combinatorics Review 1

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1. If I roll three fair 4-sided dice, what is the probability that the sum of the resulting numbers is relatively prime to the product of the resulting numbers? (BMT Discrete 2013 2)
2. How many 5-digit numbers n are there such that each digit of n is either $0, 1, 3$, or 4, and n is divisible by 15? (BMT Discrete 2014 2)
3. How many positive even numbers have an even number of digits and are less than 10,000? (OMO Winter 2012 4)
4. A wishing well is located at the point (11, 11) in the xy-plane. Rachelle randomly selects an integer y from the set $\{0, 1, \dots, 10\}$. Then she randomly selects, with replacement, two integers a, b from the set $\{1, 2, ..., 10\}$. Compute the probability that the line through $(0, y)$ and (a, b) passes through the well. (OMO Fall 13 5)
5. David has a collection of 40 rocks, 30 stones, 20 minerals and 10 gemstones. An operation consists of removing three objects, no two of the same type. What is the maximum number of operations he can possibly perform? (OMO Winter 13 9)
6. Xander takes a permutation of the numbers 1 through 2018 at random, where each permutation has an equal probability of being selected. He then cuts the permutation into increasing contiguous subsequences, such that each subsequence is as long as possible. For example, Xander would split $(3, 1, 2, 5, 4)$ into three subsequences: $(3), (1, 2, 5)$, and (4). Compute the expected number of such subsequences. (OMO Winter 2012 26)
7. A game is played with 16 cards laid out in a row. Each card has a black side and a red side, and initially the face-up sides of the cards alternate black and red with the leftmost card black-side-up. A move consists of taking a consecutive sequence of cards (possibly only containing 1 card) with leftmost card black-side-up and the rest of the cards red-side-up, and flipping all of these cards over. The game ends when a move can no longer be made. What is the maximum possible number of moves that can be made before the game ends? (OMO Fall 2012 21)
8. Dirock has a rectangular backyard that can be represented as a 32×32 grid of unit squares. He places a rock in every grid square whose row and column number are both divisible by 3. He would like to build a rectangular fence with vertices at the centers of grid squares and sides parallel to the sides of the yard such that

   (a) The fence does not pass through any grid squares containing rocks;

   (b) The interior of the fence contains exactly 5 rocks.

   In how many ways can this be done? (OMO Winter 13 21)
9. Al has the cards $1, 2, \dots, 10$ in a row in increasing order. He first chooses the cards labeled $1, 2$, and 3, and rearranges them among their positions in the row in one of six ways (he can leave the positions unchanged). He then chooses the cards labeled $2, 3$, and 4, and rearranges them among their positions in the row in one of six ways. (For example, his first move could have made the sequence $3, 2, 1, 4, 5, \dots$ and his second move could have rearranged that to $2, 4, 1, 3, 5, \dots$) He continues this process until he has rearranged the cards with labels $8, 9, 10$. Determine the number of possible orderings of cards he can end up with. (OMO Fall 13 16)

10. Tyler rolls two 4037-sided fair dice with sides numbered $1, ..., 4037$. Given that the number on the first die is greater than or equal to the number on the second die, what is the probability that the number on the first die is less than or equal to 2018? (BMT Individual Spring 2012 4)

11. A coin is flipped until there is a head followed by two tails. What is the probability that this will take exactly 12 flips? (BMT Discrete 2013 6)

12. You are playing a game in which you have 3 envelopes, each containing a uniformly random amount of money between \$0 and \$1000. At any step, you take an envelope and look at its contents. You may choose either to keep the envelope, at which point you finish, or discard it and repeat the process with one less envelope. If you play to optimize your expected winnings, your expected winnings will be $\$E$. What is $\lfloor E \rfloor$, the greatest integer less than or equal to E? (OMO Winter 2012 33)

13. Albert and Kevin are playing a game. Kevin has a 10% chance of winning any given round in the match. If Kevin wins the first game, he wins the match. If not, he requests that the match be extended to a best of 3. If he wins the best of 3, he wins the match. If not, then he requests the match be extended to a best of 5, and so forth. What is the probability that Kevin eventually wins the match? (A best of $2n+1$ match consists of a series of rounds. The first person to reach $n+1$ winning games wins the match). (BMT Individual 2014 15).

14. Real numbers a, b, c are such that $a + b - c = ab - bc - ca = abc = 8$. Find all possible values of a. (BMT Individual 2013 7)

15. A light is controlled by a magic light switch, which randomly switches on and off. At time t seconds, the switch has a $\frac{1}{2t^2}$ chance of switching wires. If the light is off at time $t = 1$, what is the probability that the light is off at time $t = 15$? (HMMT 2007 Guts 21)

16. You are tossing an unbiased coin. The last 28 consecutive flips have all resulted in heads. Let x be the expected number of additional tosses you must make before you get 60 consecutive heads. Find the sum of all distinct prime factors in x. (BMT Individual Spring 2012 8)

17. Equilateral triangle ABC is inscribed in a circle. Chord AD meets BC at E. Suppose that $DE = 2013$ and both DB and DC are integers. Compute the number of possible values of AD. (BMT Individual 2013 19)

18. There are 2018 frogs in a pool and there is 1 frog on the shore. In each time-step thereafter, one random frog moves position. If it was in the pool, it jumps to the shore, and vice versa. Find the expected number of time-steps before all frogs are in the pool for the first time. (HMMT 2018 Guts 27)

19. Let $A = \{a_1, a_2, ..., a_7\}$ be a set of distinct positive integers such that the mean of the elements of any nonempty subset of A is an integer. Find the smallest possible value of the sum of the elements in A. (HMMT 2014 Guts 24)

20. Danielle Bellatrix Robinson is organizing a poker tournament with 9 people. The tournament will have 4 rounds, and in each round the 9 players are split into 3 groups of 3. During the tournament, each player plays every other player exactly once. How many different ways can Danielle divide the 9 people into three groups in each round to satisfy these requirements? (HMMT 2010 Guts 28)