Invariants

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1 JV Warm-up

1. An 8×8 chessboard has two opposite corner squares removed. Is it possible to tile the remaining squares with 2×1 dominoes such that every square is covered and no two dominoes overlap?

   Solution. No. Every domino covers one white square and one black square, but the two removed are the same color so there are more of one color than the other.

2. The French Open is a single-elimination tennis tournament with 128 people. In each round, players are paired off to play one game, and the winner advances to the next round with the loser eliminated from the tournament. This continues until only one player remains in the tournament. How many games are played? (look for a solution that uses little arithmetic!)

   Solution. The winner loses 0 games and every other player loses exactly 1 game. There are 127 total losses, and therefore 127 total games.

3. There are 15 red marbles and 16 green marbles in a jar. Pascal removes two marbles at a time, with the following rules:

   (a) If the marbles are both green, he puts one green marble back.
   (b) If there is one marble of each colour, he puts one red marble back.
   (c) If the marbles are both red, he puts one green marble back.

   At the end, there will be one marble left. Is it always red, sometimes red, or never red?

   Solution. The number of red marbles changes by 0 or 2 each time, so there are always an odd number of red marbles. There cannot be 0 red marbles, so the last marble is always red.

2 JV Problems

1. Let $a_1, a_2, \ldots, a_{15}$ represent an arbitrary arrangement of the numbers $1, 2, \ldots, 15$. Is the product $(a_1 - 1)(a_2 - 2) \cdots (a_{15} - 15)$ always even?

   Solution. 8 of the a_i are odd and 8 of the numbers $1, 2, \ldots, 15$ are odd. There must be one ordered pair in which a_i is odd and i is odd. Their difference is even, so the product is even.

2. Alice and Bob have a large chocolate bar, in the shape of a 10×10 grid. Each turn, a player may either eat an entire bar of chocolate, or break any chocolate bar into two smaller rectangular chocolate bars along a grid line. The player who moves last loses. Who has a winning strategy?

Solution. Let b denote the turns in which a bar is broken, e the number of turns in which a bar is eat. Every time a bar is broken the number of remaining pieces increases by 1 and every time a bar is eaten the number of remaining pieces decreases by 1. Since there is initially 1 piece and the game ends when there are 0 pieces, we have $e = b + 1$. The total number of turns is $e + b = 2b + 1$ which is always odd, so Alice will always take the last turn and therefore lose. Bob has a winning strategy.

3. 127 people play in a chess tournament. Prove that at the end of the tournament, the number who have played an odd number of games is even.

   Solution. Let g_i denote the number of games played by player i. Then $\sum_i g_i$ counts every game played exactly twice, so it is twice the numbers of games played. In particular, this sum is even. Thus the sum has an even number of odd terms.

4. In a certain island there are 13 amber, 15 brown and 17 crimson chameleons. If two chameleons of different colors meet, both of them change to the third color. No other color changes are allowed. Is it possible that eventually all the chameleons have the same color?

   Solution. No. Let a, b, c be the number of each color chameleon at a given time. The invariant is that $a - b$, $a - c$, and $b - c$ are constant in mod 3. Note that none of these differences are 0. If all the chameleons were one color, then two of a, b, c would be 0, and their difference would be 0 (mod 3), which cannot happen.

5. Snow White and the seven dwarves are sitting at a round table. Snow White starts with 8 candies, and all the dwarves start with 0 candies. Each second, everyone who has at least 2 candies passes 1 candy to each of their neighbors. After how many seconds does everyone have 1 candy? Does this ever happen?

   Solution. No. We show that Snow White always has an odd even number of candies. She can only pass an even number in any round. By symmetry, her two neighbors always have the same number of candies. If one of them passes her candy at a particular second, so will the other one. Thus she can only receive an even number of candies. Her number of candies can only change by an even number, so it will always stay even.

6. A $n \times n$ chessboard has a single corner removed. For which values of n is it possible to tile the remaining squares with 3×1 triminoes such that every square is covered and no two dominoes overlap?

   Solution. Label the square (i, j) by $i+j$ (mod 3). A trimino covers $(i, j), (i+1, j), (i+2, j)$ or $(i, j), (i, j+1), (i, j+2)$ for some i, j. These have different labels, so the tiling is impossible if there aren't the same number of squares with each label. The corner that was removed has label $1 + 1 = 2$. If the opposite corner has a label other than 2, then by symmetry there were initially the same number of these two labels, but there are no fewer with label 2. Thus the opposite corner has label 2, so $2n \equiv 2 \pmod 3$, or $n \equiv 1 \pmod 3$.

   We can check that it is possible whenever $n = 3k + 1$: tile the first row and column by themselves, and the remaining is a $3k \times 3k$ board which can easily be tiled.

7. In the parliament of Armltopia, each member has at most three enemies. You have to split the parliament into two parties. In order to maintain a peaceful atmosphere, you want to make sure that each person has at most one enemy in his party. Can you do it?

Solution. Start by arbitrarily partitioning. Let Φ the number of enemy pairs within the parties. If someone has ≥ 2 enemies in his party, switch him. Φ decreases each time, so this can only be done finitely many times. At the end, we have the desired parties.

3 A really awesome challenge problem

1. There is a field that has size 10×10 meters, divided into unit 1×1 squares. In the beginning, 9 of these squares are occupied by weed. Now, every day all unit squares that are adjacent to at least two other weed squares in the morning, also contain a weed at the end of the day. Is it possible that one day the entire field will be covered with weed?[1]

4 Varsity Warm-up

1. On the Colorful island there are 3 types of chameleons - red, blue and green. There are 13 red, 15 blue and 17 green. Every time two chameleons of different colors meet, they change their color to a third one (so if red and blue meet, they both become green). Is it possible that at some point all chameleons on the island have the same color?
2. There is a field that has size 10×10 meters, divided into unit 1×1 squares. In the beginning, 9 of these squares are occupied by weed. Now, every day all unit squares that are adjacent to at least two other weed squares in the morning, also contain a weed at the end of the day. Is it possible that one day the entire field will be covered with weed?
3. There is a complete graph G on 100 vertices. You are allowed to do the following operation: choose any 4 cycle and remove your favorite edge from it. We are interested in the minimum amount of edges you can get after applying this operations several times.
   (a) Prove, that you **can** obtain 100 edges.
   (b) Prove, that you **can't** obtain 98 edges.
   (c) So, what is the minimum - 99 or 100?

5 Varsity Problems

1. There are numbers $1, 2, 3, , 19, 20$ written on the board. You are allowed to errase any two numbers a and b and write $a + b - 1$ instead. You do this operation until one number is left - what is that number?
2. There are numbers $1, , 6$ are placed on a circle in this order. During one move you can add 1 to any three consequtive ones, or from three numbers that are alternating (no two are consequtive) substruct one. Could you make all numbers equal?
3. An $m\xi n$ table is given, each entry has a real number. In one step you can choose row or column and multiply all numbers by -1. Prove, that it is possible to transform this table, so that the sum of number in every row and every column is nonegative.

[1]If you don't solve this during practice, try it at home. I am happy to discuss at any time, but I will never tell you the answer to this one. It's so satisfying if you get it on your own that I don't want to deprive you of that ☺.

4. On a 8ξ8 board in a left bottom 3ξ3 corner there are 9 stones. During one move you are allowed to take a stone and jump over any other stone on the field that is symmteric to the original field with respect to the second one. Is it possible to collect all stones in the top right 3ξ3 corner.

5. Circle is devided into 6 sectors which contain numbers $1, 0, 1, 0, 0, 0$. You can add one to any adjacent sectors. Can you make all numbers equal?

6. A natural number is written in each square of an $m\xi n$ chessboard. The allowed move is to add an integer k to each of two adjacent numbers in such a way that nonnegative numbers are obtained (two squares are adjacent if they share a common side). Find a necessary and sufficient condition for it to be possible for all the numbers to be zero after finitely many operations.

7. You have three printing machines. The first takes takes pair of numbers a and b and outputs pair $a + 1$ and $b + 1$; the second one takes pair of even numbers a and b and outputs pair $\frac{a}{2}$ and $\frac{b}{2}$; the third one takes two pairs a, b and b, c and outputs pair a, c. Also, all machines return original pairs of numbers. Starting with a pair $(5, 19)$ is it possible to get $(1, 1988)$?

8. There are 2000 white balls in the box and an infinite number of white/green and red squares. In one move you can change 2 balls from a box by the following rules: two white or two red for a green one, two green for a white and red, white and red for a green, green and red for a white. After several moves there are only 3 balls left. Prove, that at least one of them is green. Is it possible to leave only one ball in the box?