# Combinatorics

## PIE and Binomial Coefficients

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## Warm-up

Po-Shen Loh, 2013. If the letters of the word "DOCUMENT" are randomly rearranged, what is the probability that all three vowels will be adjacent? (For example, "DOEUCMNT" counts, but not "DOCUEMNT".)

Find an eight-letter English word for which the probability that all vowels are adjacent is as small as possible.

(Vocabulary challenge: try to minimize this probability over all English words.)

Original. A *crooked rook* is a chess piece I just made up that can only move either one square forward or one square to the right on each move. How many ways are there for a crooked rook to go from the bottom left square of an $8 \times 8$ chessboard to the top right square, while avoiding the $2 \times 2$ center of the chessboard?

Po-Shen Loh, 2013. A rearrangement of "DOCUMENT" with adjacent vowels corresponds to a rearrangement of "DOCMNT" (of which there are 6!) in which we replace the "O" with a rearrangement of "EOU" (of which there are 3!).

Therefore the probability that all the vowels will be adjacent is

$$\frac{6! \cdot 3!}{8!} = \frac{3!}{8 \cdot 7} = \frac{3}{28}.$$

If an eight-letter word has $k$ vowels, the probability is

$$\frac{(8-k+1)! \cdot k!}{8!} = 9 \cdot \frac{(9-k)! \cdot k}{9!} = 9 \cdot \frac{1}{\binom{9}{k}}.$$

This is minimized when $\binom{9}{k}$ is maximized, i.e. for $k = 4$ or $k = 5$. For $k = 5$, try "EQUATION"; for $k = 4$, "UNCTUOUS".

Original. We can draw the following table to count the number of ways to get to each square of the chessboard, following the rule that – since each square can only be reached from the two squares below or to the left – the number in each square is the sum of the two numbers in those squares.

$$\begin{bmatrix} 1 & 8 & 36 & 85 & 155 & 267 & 491 & 982 \\ 1 & 7 & 28 & 49 & 70 & 112 & 224 & 491 \\ 1 & 6 & 21 & 21 & 21 & 42 & 112 & 267 \\ 1 & 5 & 15 & & & 21 & 70 & 155 \\ 1 & 4 & 10 & & & 21 & 49 & 85 \\ 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This gives a final answer of 982. (Maybe I should propose this problem to the AIME people?)

- We saw last time that if $A$ and $B$ are two events,

$$\Pr[A \text{ or } B] = \Pr[A] + \Pr[B] - \Pr[A \text{ and } B].$$

(e.g. rolling two dice, $\Pr[\text{at least one } 6] = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$.)

# The Principle of Inclusion-Exclusion

## Seeing any of your events

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- More generally, if we have a whole bunch of events, to find the probability any of them happen:

1. Add the probability of each happening.
2. Subtract the probabilities for each pair.
3. Add the probabilities of each triple.
4. And so on, alternating adding and subtracting.

(e.g. for 3 dice, we get $\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right) - \left(\frac{1}{36} + \frac{1}{36} + \frac{1}{36}\right) + \frac{1}{216}$.)

# The Principle of Inclusion-Exclusion

## Avoiding all of your events

- Here's another incarnation of this. Suppose we want to find the probability none of $A, B, \ldots, Z$ occur. Then:

  1. Start with $1 - \Pr[A] - \Pr[B] - \cdots - \Pr[Z]$.
  2. Add back pairs: $+\Pr[A \text{ and } B] + \cdots + \Pr[Y \text{ and } Z]$.
  3. Subtract off triples, and so on...

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  3. Subtract off triples, and so on...
- **Don't do this** if $A, B, \ldots, Z$ are independent! In that case, it's easier to multiply together

$$(1 - \Pr[A])(1 - \Pr[B])(1 - \Pr[C])(\cdots)(1 - \Pr[Z]).$$

- This can be a shortcut for the dice example. The probability we don't get 6 on any of 4 dice is $\left(1 - \frac{1}{6}\right)^4 = \frac{625}{1296}$, so the probability of seeing at least one 6 is $1 - \frac{625}{1296} = \frac{671}{1296}$.

- This works for counting as well. Say we want to count some things that don't satisfy any of several bad conditions $B_1, B_2, B_3, \ldots$.
  1. First, count all of the things you have, ignoring bad conditions.
  2. Subtract the number of things with each bad condition.
  3. Add back the number of things with each pair of two bad conditions.
  4. As always, alternate signs and repeat.

# The Principle of Inclusion-Exclusion

## The counting version

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- For example, the number of four-digit numbers without adjacent 1's is:

$$9000 - (100 + 90 + 90) + (10 + 9 + 1) - 1 = 8719.$$

(Our bad conditions are $11xx$, $x11x$, and $xx11$.)

Original. If a random divisor of $10^{10}$ is chosen, what is the probability that it divides at least one of 1000, 1250, or 1280?

AIME, 2002. Count the number of sets $\{A, B\}$, where $A$ and $B$ are nonempty subsets of $\{1, 2, 3, \ldots, 10\}$ with no elements in common.

Po-Shen Loh, 2013. Three couples (Amy and Bob, Chad and Dana, and Emma and Fred) want to sit in a row with no couple getting adjacent seats. Find the number of ways to do this.

Folklore. Many people simultaneously take off their hats and toss them into the air. Each catches a hat, resulting in a uniformly random permutation of hats. Prove that the probability nobody has their own hat is about $\frac{1}{e} \approx 0.37$.

To figure out all the probabilities, we need two insights:

1. A random divisor $x = 2^a \cdot 5^b$ divides $1000 = 2^3 \cdot 5^3$ if $a \leq 3$ and $b \leq 3$, which has probability $\frac{4}{11} \cdot \frac{4}{11} = \frac{16}{121}$.
2. Asking that $x$ divides 1000 and 1280 is the same as asking that it divide $\gcd(1000, 1280) = 40$.

Skipping the intermediate calculations, we get

$$\left(\frac{16}{121} + \frac{10}{121} + \frac{18}{121}\right) - \left(\frac{8}{121} + \frac{8}{121} + \frac{4}{121}\right) + \frac{4}{121} = \frac{28}{121}.$$

AIME, 2002. To choose $A$ and $B$, we divide $\{1,\ldots,10\}$ into three classes: in $A$, in $B$, or not in either. There are $3^{10}$ ways to do this, but $2^{10}$ make $A$ empty, and $2^{10}$ make $B$ empty, while 1 makes both empty. So there are $3^{10}-2\cdot 2^{10}+1$ ways to choose $(A,B)$. In $\{A,B\}$, order doesn't matter, so we divide by 2 and get 28501.

Po-Shen Loh, 2013. There are 720 ways to seat the couples; but for each couple, there are $2\cdot 5! = 240$ ways to seat them next to each other; for two couples, there are $2^2\cdot 4! = 96$ ways to seat both next to each other; and finally, there are $2^3\cdot 3! = 48$ ways to seat all three couples next to each other. So altogether we get

$$720 - 3\cdot 240 + 3\cdot 96 - 48 = 240 \text{ ways.}$$

Folklore. Let $n$ be the number of people present. The $k$-wise intersections of events we want to avoid are situations where $k$ people get their own hats. For any of $\binom{n}{k}$ groups of $k$ people, there are $(n-k)!$ ways to choose the remaining hats, so the probability is $\frac{(n-k)!}{n!}$.

Multiplying $\binom{n}{k}$ by $\frac{(n-k)!}{n!}$ gives $\frac{1}{k!}$, so this is what the $k$-th term of the inclusion-exclusion sum looks like. So the probability we want is

$$1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots\pm\frac{1}{k!}\pm\cdots\pm\frac{1}{n!}.$$

This is the first $n$ terms of the sum $\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}$, which is known to converge to $\frac{1}{e}$ (in general, $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$; here, $x=-1$).

# Binomial coefficients

If we have $n$ distinct objects, the number of ways to choose a set of $k$ of them is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, pronounced "$n$ choose $k$".

Key facts:

1. $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + y^n$.
2. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
3. $\binom{n}{2} = \frac{n(n-1)}{2}$ is particularly useful. It's the number of ways to choose pairs of objects. Also, $1 + 2 + \cdots + n = \binom{n+1}{2}$.
4. $\binom{n}{k}2^{-n}$ is the probability that when $n$ fair coins are flipped, $k$ of them come up heads.
5. A bar graph of the numbers $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ forms a "bell curve" in which the middle numbers take up most of the bulk.

# The crooked rook problem

We can solve the crooked rook problem using binomial coefficients.

We will denote the bottom left corner of the board by $(1, 1)$, and the top right corner by $(8, 8)$; the two allowed steps are $(+1, +0)$ and $(+0, +1)$, and the center $2 \times 2$ square consists of the points $\{(4, 4), (4, 5), (5, 4), (5, 5)\}$.

In total, there are $\binom{14}{7}$ ways to get from $(1, 1)$ to $(8, 8)$: it takes 14 steps to do so, and any 7 of them can be $(+1, +0)$ steps while the remainder are $(+0, +1)$.

Paths through the center $2 \times 2$ square must visit either $(4, 5)$ or $(5, 4)$, but not both. There are $\binom{7}{3}$ ways to get from $(1, 1)$ to $(4, 5)$, and $\binom{7}{3}$ ways to get from $(4, 5)$ to $(8, 8)$, so $\binom{7}{3}^2$ of the paths go through $(4, 5)$ and must be discarded. Discarding paths through $(5, 4)$ similarly, we get a final answer of $\binom{14}{7} - 2 \cdot \binom{7}{3}^2 = 982$.

# Pirates and Gold

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In a common variant, you allow distributions where some pirates get no gold at all. This is equivalent to splitting $n+k$ pieces of gold so that all pirates get at least one piece, and then taking a piece of gold from each. So it can be done in $\binom{n+k-1}{k-1}$ ways.

Original. Find the number of zeroes at the end of $\binom{200}{100}$.

Lovász et. al., *Discrete Math*. In how many ways can you distribute $n$ pennies to $k$ children if each child must get at least 5?

Po-Shen Loh, 2013. Call a sequence of letters *increasing* if the letters in it appear in alphabetical order (e.g. "BOOST" or "AEGIS"). How many increasing sequences of 52 letters are there?

Folklore. Prove that the central binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$ is between $\frac{2^n}{n+1}$ and $2^n$. (A good estimate, surprisingly, is $\sqrt{\frac{2}{\pi n}} \cdot 2^n$.)

Original. To find the number of zeroes at the end of 100!, we count powers of 5: there are 20 numbers between 1 and 100 divisible by 5, and 4 of them are divisible by 25, so we get 24. (There are lots more powers of 2, so we don't worry about those.) Similarly, there are 49 zeroes at the end of 200!. So the answer is $49 - 24 - 24 = 1$.

Lovász et. al. We can begin by giving 4 pennies to each child; then the number of ways to distribute the remaining $n - 4k$ pennies so that each child gets at least one more is $\binom{n-4k-1}{k-1}$.

Po-Shen Loh, 2013. The increasing sequence is determined uniquely by the letters it contains. To pick letters, we split the 52 "slots" between the 26 letters in the alphabet, which can be done in $\binom{52+26-1}{26-1} = \binom{77}{25}$ ways.

Folklore. Consider the sum

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\lfloor n/2 \rfloor} + \cdots + \binom{n}{n} = 2^n.$$

There are $n+1$ terms. Each is positive, so each must be less than $2^n$. On the other hand, $\binom{n}{\lfloor n/2 \rfloor}$ is the largest term; so it must be at least the average of all the terms, which is $\frac{2^n}{n+1}$.