Combinatorics

Induction, pigeonhole, and **BRUTE FORCE**

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Proof techniques

Induction.

Given 100 math problems, you can solve the first one. If you can solve the n-th problem, you can solve the $(n+1)$-th. Therefore you can solve all the problems.

The pigeonhole principle.

More than 10 million people live in New York. Nobody has more than 500 000 strands of hair on their heads. Therefore there are 20 people living in New York with the exact same number of strands of hair.

Of more practical use is the following: if you have 2 colors of socks in your sock drawer, and pull out 3 socks, then 2 will have the same color.

Proof techniques

Problems

1. Prove by induction that the sum of the first n odd numbers is n^2.
2. Given a set of 51 integers between 1 and 100, prove that:
   - Two of them are adjacent;
   - One of them divides another;
   - We can choose a nonempty subset of them whose sum is divisible by 50.
3. Prove that the Fibonacci numbers defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ satisfy $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.
4. Prove that if a 3×7 grid of points is colored red and blue, there will always be a rectangle whose 4 corners have the same color.

Proof techniques
Solutions

1. $1 = 1^2$, and if $1 + 3 + \cdots + (2n - 1) = n^2$, then $1 + 3 + \cdots + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$.
2. Divide $\{1, 2, \ldots, 100\}$ in the following groups:
   - $\{1, 2\}, \{3, 4\}, \ldots, \{99, 100\}$.
   - $\{1, 2, 4, \ldots, 64\}, \{3, 6, 12, \ldots, 96\}, \{5, 10, \ldots, 80\}, \ldots, \{99\}$.

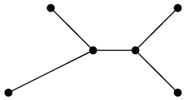
   For the third part, if the integers are $a_1, \ldots, a_{51}$, define $S_n = a_1 + \cdots + a_n$. Among $S_1, \ldots, S_{51}$, two are the same mod 50, so $S_i - S_j = a_{j+1} + \cdots + a_i \equiv 0 \pmod{50}$.
3. We have $F_1 + F_2 = 2 = F_4 - 1$. If $F_1 + \cdots + F_n = F_{n+2} - 1$, then $F_1 + \cdots + F_{n+1} = F_{n+2} + F_{n+1} - 1 = F_{n+3} - 1$.
4. Each column of 3 points contains 2 of the same color. These can be in 6 positions; 2 of the 7 columns will have them in the same positions, forming a rectangle.

Counting by

BRÜTE FÖRCE

Classical problems.

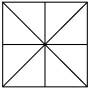
1. How many ways are there to write 9 as a sum of positive integers (e.g. $9 = 4 + 3 + 2$) if order does not matter?
2. How many ways are there to split $\{1, 2, \ldots, 6\}$ into 3 groups?
3. A graph is a set of vertices, some of which are connected by edges. A tree is a graph with n vertices and $n - 1$ edges which is connected: there is a path between any two vertices.



How many different trees with 6 vertices are there?

More counting by BRÜTE FÖRCE

Personal experience. The 8 triangles in the figure below are colored red and blue. Up to rotations, how many distinct colorings are possible?



Duke Math Meet 2008. A chess knight can jump from a point (x, y) to the 8 points $(x \pm 1, y \pm 2)$ and $(x \pm 2, y \pm 1)$. In how many ways can it get from $(0, 0)$ to $(2, 2)$ in 4 jumps?

ARML 2006. A graph on 5 vertices is chosen by adding each of the 10 edges with a probability of $\frac{1}{2}$. What is the probability that the graph is connected?

Answers to counting by BRÜTE FÖRCE

1. 30 (29 if you don't count the "sum" $9 = 9$).
2. 90.
3. 6.
4. 70.
5. 54, I think.
6. Probably $\frac{91}{128}$.