

Lattice Paths and Friends

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1 Lattice Paths

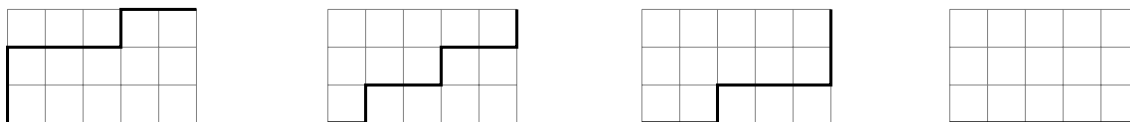


Figure 1: Some lattice paths from $(0, 0)$ to $(5, 3)$

- (Moving Up and Right) For this problem, every step is either one unit to the right or one unit up. (Note: a “way” to go from a to b is a sequence of moves, eg. UURURRR)
 - How many ways are there to go from $(0, 0)$ to $(2, 3)$?
 - How many ways are there to go from $(1, 1)$ to $(4, 7)$?
 - In general, how many ways are there to from $(0, 0)$ to (m, n) ?
- (Moving on the Grid) For this problem, every step is either one unit up, down, left, or right.
 - How many ways are there to go from $(0, 0)$ to $(4, 3)$ using exactly one left and no downs?^[1]
 - How many ways are there to go from $(0, 0)$ to $(3, 5)$ in exactly 10 moves? 11 moves? 12?
 - How many ways are there to go from $(0, 0)$ to $(4, 3)$ using exactly one left and no downs when you must stay within the rectangle (i.e. at any point (x, y) in your path, $0 \leq x \leq 4$ and $0 \leq y \leq 3$)?
- (Flying along the path) You have acquired a jet pack which allows you to go 2 units right (it’s picky and only goes right). For this problem, every step is either one unit right, one unit up, or two units right (using your jetpack).
 - How many ways are there to go from $(0, 0)$ to $(7, 3)$ if you only have fuel for one use of the jetpack?
 - If you have unlimited fuel, how many ways are there to go from $(0, 0)$ to $(7, 3)$?
- (A hole in the middle) For this problem, every step is either one unit right or one unit up.
 - How many ways are there to go from $(0, 0)$ to $(5, 3)$ avoiding $(1, 1), (1, 2), (2, 1), (2, 2)$?

2 Other Classical Counting Problems

- How many unique ways are there to rearrange the letters in LATTICEPATHS?
- Annie, Batman, C.J., David, and Emily get on an elevator in Wean hall on the 1st floor. The elevator opens on each floor as it rises to the 8th floor (so there are 8 places where people may get off). How many ways are there for them to be anti-social (all get off at different floors)?

^[1]Hint: How many steps do you need to take?

- Using only the digits 1 through 6, with repetition allowed, how many ways are there to make a 6-digit number divisible by 5? Divisible by 4? ^[2]
- It is the future, and you are now old. So, you must be the one providing Halloween candy to the children that come to your door. You have bought 22 candies, and 8 kids come to the door. You may not eat any candy yourself. How many ways can you distribute the candy to the children? How many ways can you distribute the candy if the parents glare at you until you give each child at least 1 piece?
- Misha has become the supreme overlord of Earth, and has mandated that all decks of cards must now come with a 5th suit - the Squirrel suit (so a deck now has 65 cards, with 5 suits of 13 cards each). Calculate the proper ranking of poker hands for these new decks (i.e, rank the number of distinct pairs, triples, full houses, 4-of-a-kinds, 5-of-a-kinds, straights, flushes, and straight flushes there are in such a deck).

3 Challenge Problems

- A palindrome on the alphabet $\{1, 0\}$ is a sequence of 1's and 0's which reads the same from left to right as it does from right to left. Let $P(n)$ denote the number of palindromes of length n over $\{1, 0\}$. For how many values of n is $1000 < P(n) < 10000$?
- Let there be 36 people in a class. Every day, we split these people into 6 groups of 6, where no 2 people are allowed to be in the same group more than once. How many times can we do this split? If we have 100 people instead, splitting into 25 groups of 4, how many times can we split?

4 Background

- There are $n! = n(n-1) \cdots 2 \cdot 1$ ways to arrange the integers 1 through n .
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ counts the number of ways to pick k things from n distinguishable things without order (eg. the number of ways to arrange k *'s and $n-k$ |'s).
- If you can split what you are counting into nonoverlapping cases which cover all possibilities, you can add up the results you get for each of these cases to get a result for the entire thing.
- If you have a fixed process of steps to decide how to pick your objects (eg. pick a card by picking a suit, 4 ways, and a rank, 13 ways), and you know the number of ways to accomplish each step, you can multiply the numbers of ways per step to get the total number of ways to complete the process (so in this case we have $4 \cdot 13 = 52$ ways to pick a card).
- If you can count the number of "bad" ways to do something, then subtracting that from the total number of ways to do that thing will give you the number of "good" ways to do that thing. (This is often called complementary counting).
- It's often good to count the same thing in multiple ways to check your answer!

^[2]Hint: A number is divisible by 4 if and only if the last 2 digits of that number are also divisible by 4

5 Selected Solutions

5.1 Lattice Paths

To any path, we associate a string of letters U,R,L,D in accordance

- There are $\binom{5}{3} = 10$ ways to order 3 'R's and 2 'U's.
 - You need 3 'R's and 6 'U's, so there are $\binom{9}{3} = 84$ paths.
 - There are $\binom{m+n}{n}$ ways to order m 'R's and n 'U's.
- We want all strings of 1 'L', 3 'U's and 5 'R's for a total of $\binom{9}{1,3,5} = \frac{9!}{1!3!5!} = 504$ paths.
 - To get to (3,5) takes 8 steps right or up, and for each step left or down, we need a matching step up or right. Thus for 10 steps in total, we have 2 extra steps, or one extra step left/down plus the matching step right/up. There are $1260 = \binom{10}{1,5,4}$ paths with one step left (1 'L', 5 'U's, 4 'R's), and with one step down there is 1 'D', 3 'R's and 6 'U's, and there are $\frac{10!}{1!3!6!} = 840$. Adding together, we have 2100 paths in total.

There are no paths of length 11! (not factorial)

For a path of length 12, we need a total of 2 'L' and 'D' steps. There are three ways to do this:

-Two 'L's: 2x 'L', 5 x 'R', 5 x 'U' gives $\frac{12!}{2!5!5!} = 16632$.

-One 'L' one 'D': 1 x 'L', 4 x 'R', 1 x 'D', 6 x 'U' gives $\frac{12!}{1!4!1!6!} = 27720$.

-Two 'D': 3 x 'R', 2 x 'D', 7 x 'U' gives $\frac{12!}{3!2!7!} = 7920$.

Adding together we have 52272 total paths.

- If we don't use the jetpack, there are $\binom{10}{7} = 120$ paths from (0,0) to (7,3) by question 1. If we use the jetpack, we replace two 'R's by a single 'J' (Jetpack), so we have 3 x 'U', 5 x 'R', 1 x 'J'. There are $\frac{9!}{3!5!1!} = 504$ such paths, for a total of 624.
 - With unlimited jetpack uses, we can also have paths with 2 or 3 jetpack uses.
- The only ways to avoid the square are by going right 3 times or up 3 times for our first 3 moves. With 3 rights: 2 rights and 3 ups remain for a total of $\binom{5}{2} = 10$ paths. With 3 ups: 5 rights and 0 ups remain for a total of $\binom{5}{0} = 1$ path. Summing these, we have a total of $10 + 1 = 11$ paths avoiding the square. (Try doing this with complementary counting!)

5.2 Classic counting problems

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