Pigeonhole Exercise

1. Warm-up

1. Students from three classes are going to camping. What is the minimum number of students coach should pick to set a tent such that there are three students from the same class for sure?
2. 100 books are delivered to students. What is the mazimum number of students could be so that every student got different number of books?

2. Divisibility

3. Prove, that there is a number consisting only of ones that is divisible by 2017.
4. Prove, that there is a number that ends on 2016 and is divisible by 2017.
5. Is it possible to find a power of 3 that ends as $\ldots 0001$?
6. The sum of 100 positive integers (all of them are at most 100) is equal to 200. Prove, that some of those numbers sum up to 100.

3. Additional arguments

7. 8 different positive integers are given, all at most 15. Prove, that among all pairwise differences of those numbers at least three are the same.
8. Ten students composed 35 problems for PUMaC. Some of them came up with 1, 2 and 3 problems. Prove, that there is a student who came up with at least five problems.
9. Is it possible to write down 57 two-digit positive integers such that no two add up to 100?
10. 60 people come to a Sunday ARML practice (could we dream a little bit?). It happens that among any ten students there are at least three from the same school. Prove, that there are at least 15 students from the same school on practice.
11. $n+1$ positive integers that are at most $2n$ are given. Prove, that there exist three numbers such that two of them sum up to the third one.

4. Geometry

12. Alex starts with a square and cut it (along straight lines) into two peaces per move and then repeat. Prove, that at some point it is possible to find 100 peaces that will have the same number of vertices.
13. 7 segments are given, all have leght from 0.1m to 1m. Prove that there are three of them that can form a triangle.
14. Prove that one cannot pick more than 5 points inside a circle of radius one such that all pairwise distances are grater that 1.

15. In a square with side length 5cm 126 points are marked. Prove that you can draw a circle of radius 1 such that it will contain at least 6 points.

16. There are 1937 trees are grown inside a circular field (of radius 215). Prove, that there are two trees with distance less than 10.

17. There are 25 points marked on the plane such that among any three of them there is a pair with distance at most 1. Prove, that you can draw a circle of radius 1 that covers at least 13 points.

5. Optimization

18. What is the maximum number of colours that can be used to paint an 8×8 chessboard such that every square is painted in a single colour, and is adjacent, horizontally, vertically, or diagonally, to at least two other squares of its own colour?

19. What is the largest number of knights that can be placed on an 8×8 chessboard so that no two knights are attacking each other?

20. What is the maximum number of integers could be chosen among $\{1, 2, \ldots, 2017\}$ such that for any two numbers their sum is not divisible by their difference?

21. A certain province issues license plates consisting of six digits (from 0 to 9). The province requires that any two license plates differ in at least two places. (For instance, the numbers 027592 and 020592 cannot both be used.) Determine, with proof, the maximum number of license plates that the province can use.

22. n points are marked inside an equilateral triangle with side length 1 in such a way that no two marked points are within distance $\frac{1}{3}$ of each other. What is the largest possible value for n?

23. What is the maximum number of points that could be placed inside rectangle 3×4 such that the distances between point are greater than $\sqrt{5}$?

24. (a) Let A be the largest subset of $\{1, 2, \ldots, n\}$ such that for each $x \in A$, x divides at most one other element of A. Prove that

$$\frac{2n}{3} \leq |A| \leq \lceil \frac{3n}{4} \rceil$$

(b) Let B be the largest subset of $\{1, 2, \ldots, 2010\}$ such that B neither contains two elements one of which divides the other, nor contains two elements which are relatively prime. What is $|B|$?