Pigeonhole Introduction

1. Warm-Up

1. During last week Alex ate 15 chocolate bars. Prove, that during one of the days he ate at least three chocolate bars.

2. There are 82 colored cubes. Prove that you can either pick ten cubes of different colors or ten cubes of the same color.

3. 7 rays are drown on the plane from the origin. Prove, that some pair of rays form an angle that is greater than $51°$.

4. Is it possible to distribute 100 nuts between 15 students so that every student get different amount of nuts?

2. Are you warm enough?

5. There are 7 studens in class and their total age is 332 years. Prove, that there are three students in class with total age at least 142 years.

6. Prove that among any n integers you can pick some of them so that their sum is divisible by n.

7. Nine points with integer coordinates are marked in the space. Prove, that one of the segments that connects some of these points passes through a lattice point.

3. Optimization

This problems will ask you to find the minimum or maximum of something subject to some conditions. For example - what is the minimum amount of time you may spend to solve all of these problems?

8. A square grid of 16 dots (4×4) contains the corners of nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of 14 squares whose sides are parallel to the sides of the grid. What is the smallest possible number of dots you can remove so that, after removing those dots, each of the 14 squares is missing at least one corner?

9. There are some number of molecules are floating inside an equilateral triangle with side length 1. If any two molecules are at distance $\frac{1}{2}$ or closer, then the force between them is strong enough to bring them together. Then they become a one molecules. What is the largest possible number of different molecules could be floating at the end?

10. Let A be the largest subset of $\{1, 2, \ldots, 100\}$ such that A does not contain two coprime elements. What is $|A|$?

11. We say that two different cells of an 8×8 board are neighbouring if they have a common side. Find the minimal number of cells on the 8×8 board that must be marked so that every cell (marked or not marked) has a marked neighbouring cell.

4. Bonus

12. Among a group of 120 people, some pairs are friends. A *weakquartet* is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets?

Solutions

1. By the pigeonhole principle, one day he ate at least $\lceil\frac{15}{7}\rceil = 3$ chocolate bars.

2. Suppose there are not ten cubes of any one color. Then there are at most 9 cubes of each color, so there are at least $\lceil\frac{82}{9}\rceil = 10$ different colors.

3. Consider the 7 pairs of consecutive rays. The angles between them form a full circle, so there are 7 angles that add to 360°. The largest of these angles must be at least $\lceil\frac{360}{7}\rceil^\circ = 52^\circ$.

4. If the students each get a different number of nuts, there must be at least $0+1+2+3+\cdots+14 = \frac{14\cdot 15}{2} = 105$ nuts. Since there are only 100 nuts, it is impossible for everyone to get a different number.

5. The sum of the ages of the three oldest students is at least $\lceil\frac{3}{7}\cdot 332\rceil = 143$.

6. Let the integers be $a_1, a_2, \ldots, a_n$. Consider the numbers $b_k = \sum_{i=1}^{k} a_i$, $k \in [n]$. If some b_k is congruent to 0 (mod n), then this gives a choice of a_i's whose sum is divisible by n. Otherwise, all b_k are congruent to a number in $\{1, 2, \ldots, n-1\}$ (mod n). By the pigeonhole principle, there are $j < k$ such that b_j and b_k are equivalent in mod n. Then $a_{j+1} + a_{j+2} + \cdots + a_k = b_k - b_j \equiv 0 \pmod{n}$, which gives the desired choice of a_i's.

7. For a point (a, b, c) with integer coordinates, there are 8 possible ordered triples $(a(\text{mod } 2), b\,(\text{mod } 2), c\,(\text{mod } 2)$. By the pigeonhole principle, two of our 9 points are equivalent (mod 2) in each of their coordinates. The midpoint of these points has integer coordinates, and lies on the line segment between these two points.

8. There are four 2×2 squares, and no two of them share any corners, so we must remove at least 4 points. If the points are $\{(x, y) : x, y \in [4]\}$, we can remove the points $(1, 1), (2, 3), (3, 2), (4, 4)$, and check that there are no squares remaining. Thus, the minimum is 4 points.

9. The maximum is 4. We can have achieve this by having one molecule at each vertex, and one at the center of the triangle. To see that we can have at most 4, divide the triangle into four triangles of side length $\frac{1}{2}$ by connecting the midpoints of the sides. We can have at most one point from each of these four triangles.

10. The maximum is 50.

    n and $n+1$ are always coprime, so A can contain at most one element from each of $\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{99, 100\}$. There are 50 sets, so $|A| \leq 50$.

    The example where $|A| = 50$ is the set of all the even numbers.

11. The minimum is 20.

    Imagine this grid is colored like a chess board. White squares only have black neighbors and black squares only have white neighbors. It suffices to show that the minimum number of black squares we can mark so that every white square has a marked neighbor is 10. Let the white squares be those whose coordinates have an even sum.

To see that 10 is enough, you can check that if you mark the ten black squares $(2,1),(6,1),(4,3),(8,3),(1,4),(4,5),(6,5),(2,7),(8,7),(5,8)$, then each white square has a marked neighbor.

One way to see that 10 squares are needed is to find 10 white squares such that no two of them share a common neighbor. One example of how to do this is the ten squares $(1,1),(1,5),(2,8),(3,3),(4,6),(5,1),(6,4),(6,8),(8,2),(8.6)$.