Coloring

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Problems

0. Starting with a $k \times m$ rectangular grid, we select n random squares. Show that we can always find $n/4$ of them such that no two that we select share an edge or a corner.

1. A checker starts in the bottom-left corner of an 8×8. Each step it goes to a neighboring cell (two cells are neighbors if they have a common side). Can the checker be in the top-right corner after 2018 steps ?

2. Now the checker starts in the bottom-left corner of a 101×101 grid. Find the number of possible cells that checker can be after 100 steps.

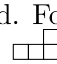
3. A bug starts in the bottom-left corner of an 8×8 chess table. At every step it can move one step in three directions: up, right, or diagonally down and left. Is there a path in which the bug visits every cell exactly once?

4. We have some number of 2×2 tiles and some number of 1×4 tiles. We can use all the tiles to tile an $m \times n$ rectangle. If one of the 2×2 tiles breaks and we replace it with a 1×4 tile, can we still tile the rectangle?

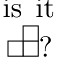
5. In a 5×5 chess table each cell has a + or a -. Initially, one cell has a - and all other cells have a +. At each step we can pick any $a \times a$ $(a > 1)$ subtable and flip all the signs in that subtable. If the - starts in an arbitrary cell, can we eventually make every cell have a +?

6. A 7×7 grid is tiled by a single 1×1 tile and sixteen 1×3 tiles. Find all possible positions of the 1×1 tile.

7. In a 3×3 chess table we have black knights in top left and right cells and white knights in bottom left and right cells. Prove that we need at least 16 steps to make black knights in bottom corners and white knight in top corners. (Each move of a knight goes two steps in one direction, and one step in a perpendicular direction).

8. A $n \times n$ table has all four corner cells removed. For which values of n we can tile this table by L-tetraminos? An L-tetramino looks like: , and can be rotated and/or flipped.

9. (IZHO 2007) You are given 111 and must place them on the cells of an $n \times n$ grid (one unit cell may contain one coin, many coins, or may be empty), such that the difference between the number of coins from two neighboring cells (that have a common edge) is 1. Find the maximal n for which this is possible.

10. (IMO Shortlist 2002) For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. For which values of n is it possible to cover all the black squares with non-overlapping trominos shaped like: ? When it is possible, what is the minimum number of trominos needed?

1 Selected Solutions

For problems with chess tables/checker tables/rectangular grids, we consider the squares being colored black and white in a checkerboard pattern, with black in the bottom-left corner.

0. Group ('color') the squares according to the parity of each of their two coordinates. No two squares of the same group share an edge or a corner. By the pigeonhole principle, one of the groups must have at least $n/4$ squares in it.

1. Yes, it can be on any black square.

2. The checker can be on any black square on or below the northwest-southeast main diagonal. By splitting the black squares up into parallel diagonals, we see that there are $1+3+5+\cdots+101$ squares. Using the formula $\sum_{i=1}^{k} 2i-1 = k^2$ we have that $1+3+5+\cdots+101 = \sum_{i=1}^{51} 2i-1 = 51^2 = 2601$.

3. No. Coloring the northwest-southeast diagonals alternately red, blue, green, red, blue, green etc. the bug always steps from red to blue, blue to green, and green to red. There are 21 red, 22 blue, and 21 green squares. Since we start on a red square, we can only reach 21 blue squares without repeating a red square.

4. No. It suffices to show that for a fixed (m, n), in any such tiling, the number of 2×2 tiles has fixed parity. In particular, if m, n are both congruent to 2 (mod 4), then we need an odd number of 2×2 tiles, and otherwise we need an even number.

   The key idea is to color the northwest-southeast diagonals by 4 alternating colors, red, blue, green, yellow. Each 1×4 covers one square of each color. A 2×2 square colors two of one color, one of each of the adjacent colors, and none of the non-adjacent color.

   If one of the dimensions is a multiple of 4, there are equally many of the four colors. Thus there must be equally many 2×2 tiles that cover 2 red and 0 green as tiles that cover 0 red and 2 green, and likewise with blue and yellow. Thus there are evenly many tiles. If neither dimension is a multiple of 4, there are again equally many red and green, but two more blue than yellow. Thus we need one more tile that covers 2 blue and 0 yellow than 0 blue and 2 yellow, so the total number of 2×2 tiles is odd.

5. Omitted, for now.

6. Coloring northwest-southeast diagonals of the board alternately red, white, blue, each 3×1 covers one square of each color. If red is in the bottom-left corner, there are 17 red squares, 16 white squares, and 16 blue squares, so the 1×1 must be in a red square. Repeating this with the northeast-southwest diagonals, the 1×1 must be in a square that is colored red both ways. These squares are all those such that both coordinates are chosen from $\{1, 4, 7\}$.

7. Omitted, for now.

8. Omitted, for now.

9. The number of coins in white cells must have one parity, and in black cells the opposite parity. There must be an odd number of cells with odd parity, so n is odd. The cells with odd parity have at least one coin, so $\frac{n^2-1}{2} \leq 111$. The greatest such number is $n = 13$.