Generating Functions I

Misha Lavrov

ARML Practice 5/4/2014

Warm-up

Problems

1. (ARML 2010 T-6.) Compute the number of 4-letter “words” (sequences of 4 letters, whether or not they appear in the dictionary) containing at least two E’s.

2. A nonstandard die has the following six faces.

   If three such dice are rolled, what is the probability of getting a total of 7?

We will solve problem # 1 in the following unusual way: consider the polynomial

$$\Psi = (A + B + C + \cdots + Y + Z)^4.$$

If we expand Ψ, we get the sum

$$\Psi = AAAA + AAAB + AAAC + \cdots + ZZZY + ZZZZ.$$

In other words, Ψ is the sum of all 4-letter words.

We only care about whether a given letter is E or not, so define

$$\begin{aligned}\Upsilon &= (N + N + N + N + E + N + N + \cdots + N + N)^4 \\ &= (E + 25N)^4 \\ &= E^4 + 100E^3N + 3750E^2N^2 + 62500EN^3 + 390625N^4.\end{aligned}$$

We only care about whether a given letter is E or not, so define

$$\begin{aligned}\Upsilon &= (N+N+N+N+E+N+N+\cdots+N+N)^4\\ &= (E+25N)^4\\ &= E^4+100E^3N+3750E^2N^2+62500EN^3+390625N^4.\end{aligned}$$

Each word in Ψ became a word in Υ, but we have forgotten some information; for example, "MEEP" became "NEEN" and then N^2E^2. The words with at least two E's became either E^4 or E^3N or E^2N^2, so there are $1+100+3750=3851$ such words.

Warm-up

Solution to Problem # 2

We can do the same thing in the second problem. The outcomes of a single die are given by the polynomial

$$\Delta = \text{⚀} + \text{⚀} + \text{⚁} + \text{⚁} + \text{⚁} + \text{⚂}$$

and the outcomes of three dice rolls are given by

$$\Delta^3 = \left(\text{⚀} + \text{⚀} + \text{⚁} + \text{⚁} + \text{⚁} + \text{⚂} \right)^3 .$$

Warm-up

Solution to Problem # 2

We can do the same thing in the second problem. The outcomes of a single die are given by the polynomial

$$\Delta = ⚀ + ⚀ + ⚁ + ⚁ + ⚁ + ⚂$$

and the outcomes of three dice rolls are given by

$$\Delta^3 = \left(⚀ + ⚀ + ⚁ + ⚁ + ⚁ + ⚂ \right)^3 .$$

However, these outcomes are in a form like ⚂⚀⚂ or ⚁⚁⚁. All we care about is the total value of the three dice. How do we get the total out of such a product?

Warm-up

Solution to Problem # 2

We “forget some details” by making the substitutions ⚀ $= x$, ⚁ $= x^2$, and ⚂ $= x^3$. Then ⚂⚀⚂ becomes $x^3 \cdot x \cdot x^3 = x^7$: the power of x is the total value of the dice.

With this substitution we get the function $F(x) = (2x + 3x^2 + x^3)^3$ in place of Δ^3. We can expand $F(x)$ to get

$$F(x) = x^9 + 9x^8 + 33x^7 + 63x^6 + 66x^5 + 36x^4 + 8x^3.$$

Warm-up

Solution to Problem # 2

We "forget some details" by making the substitutions ⚀ $= x$, ⚁ $= x^2$, and ⚂ $= x^3$. Then ⚂⚀⚂ becomes $x^3 \cdot x \cdot x^3 = x^7$: the power of x is the total value of the dice.

With this substitution we get the function $F(x) = (2x + 3x^2 + x^3)^3$ in place of Δ^3. We can expand $F(x)$ to get

$$F(x) = x^9 + 9x^8 + 33x^7 + 63x^6 + 66x^5 + 36x^4 + 8x^3.$$

The number of dice outcomes with a value of 7 is the coefficient of x^7, which I will write $[x^7]F(x) = 33$.

So the probability of getting a total of 7 is $\frac{33}{216} = \frac{11}{72}$.

Infinitely many options

(ARML 1995 T-3.[1]) Compute the number of ways in which 45 one-dollar bills can be distributed to 7 people so that no person receives less than \$5.

[1] I slightly changed the numbers.

Infinitely many options

(ARML 1995 T-3.[1]) Compute the number of ways in which 45 one-dollar bills can be distributed to 7 people so that no person receives less than \$5.

Let $G(x) = (x^5 + x^6 + x^7 + x^8 + \cdots)^7$. A term in the expansion of $G(x)$ might look like $x^6 x^{15} x^5 x^{105} x^{12} x^7 x^5$. We think of this as encoding a distribution of money in which the 7 people receive \$6, \$15, \$5, \$105, \$12, \$7, and \$5.

[1] I slightly changed the numbers.

Infinitely many options

(ARML 1995 T-3.[1]) Compute the number of ways in which 45 one-dollar bills can be distributed to 7 people so that no person receives less than \$5.

Let $G(x) = (x^5 + x^6 + x^7 + x^8 + \cdots)^7$. A term in the expansion of $G(x)$ might look like $x^6 x^{15} x^5 x^{105} x^{12} x^7 x^5$. We think of this as encoding a distribution of money in which the 7 people receive \$6, \$15, \$5, \$105, \$12, \$7, and \$5.

But of course when simplifying $G(x)$, we don't stop there. The term above gets simplified to $x^{6+15+5+105+12+7+5} = x^{155}$. All the information we have left is that the total amount of money we've given out is \$155.

[1] I slightly changed the numbers.

Infinitely many options

(ARML 1995 T-3.[1]) Compute the number of ways in which 45 one-dollar bills can be distributed to 7 people so that no person receives less than \$5.

Let $G(x) = (x^5 + x^6 + x^7 + x^8 + \cdots)^7$. A term in the expansion of $G(x)$ might look like $x^6 x^{15} x^5 x^{105} x^{12} x^7 x^5$. We think of this as encoding a distribution of money in which the 7 people receive \$6, \$15, \$5, \$105, \$12, \$7, and \$5.

But of course when simplifying $G(x)$, we don't stop there. The term above gets simplified to $x^{6+15+5+105+12+7+5} = x^{155}$. All the information we have left is that the total amount of money we've given out is \$155.

This means that we can answer the ARML question by computing $[x^{45}]G(x)$.

[1] I slightly changed the numbers.

Some fancy algebra

Using the formula for the sum of a geometric series, we can write

$$G(x) = \left(\frac{x^5}{1-x}\right)^7 = \frac{x^{35}}{(1-x)^7}.$$

The coefficient of x^{45} in $G(x)$ is the coefficient of x^{10} in $\frac{1}{(1-x)^7}$.

Some fancy algebra

Using the formula for the sum of a geometric series, we can write

$$G(x) = \left(\frac{x^5}{1-x}\right)^7 = \frac{x^{35}}{(1-x)^7}.$$

The coefficient of x^{45} in $G(x)$ is the coefficient of x^{10} in $\frac{1}{(1-x)^7}$.

Using the binomial theorem to expand $(1-x)^{-7}$, we get

$$\frac{1}{(1-x)^7} = 1 - \binom{-7}{1}x + \binom{-7}{2}x^2 - \binom{-7}{3}x^3 + \cdots .$$

So the solution is $\binom{-7}{10}$. How do we compute it?

Some fancy algebra

Using the formula for the sum of a geometric series, we can write

$$G(x) = \left(\frac{x^5}{1-x}\right)^7 = \frac{x^{35}}{(1-x)^7}.$$

The coefficient of x^{45} in $G(x)$ is the coefficient of x^{10} in $\frac{1}{(1-x)^7}$.

Using the binomial theorem to expand $(1-x)^{-7}$, we get

$$\frac{1}{(1-x)^7} = 1 - \binom{-7}{1}x + \binom{-7}{2}x^2 - \binom{-7}{3}x^3 + \cdots .$$

So the solution is $\binom{-7}{10}$. How do we compute it?

$$\binom{-7}{10} = \frac{-7 \cdot -8 \cdot -9 \cdot \cdots \cdot -16}{10!} = \frac{16!}{6!\,10!} = \binom{16}{6}.$$

Exercises

1. 1.1 Compute $\binom{-1}{5}$, $\binom{-2}{6}$, and $\binom{-3}{7}$.

   1.2 Show that $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$.

   1.3 Simplify the expression $\binom{-1/2}{n}$.

2. 2.1 Write an expression for $G(x)$ such that the coefficient of x^n, $[x^n]G(x)$, is the number of ways to give n one-dollar bills to 7 people, if each person can receive at most \$3.

   2.2 Write an expression for $G(x)$ such that $[x^n]G(x)$ is the number of ways to give n one-dollar bills to 7 people, if the number 4 is unlucky, and therefore nobody may be given exactly \$4.

Solutions

1.
   1.1 **In general,** $\boldsymbol{\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}}$, so $\binom{-1}{5} = -\binom{5}{5} = -1$, $\binom{-2}{6} = \binom{7}{6} = 7$, and $\binom{-3}{7} = -\binom{9}{7} = -36$.
   1.2 The coefficient of x^n in $\frac{1}{(1-x)^2}$ is
   $$(-1)^n \binom{-2}{n} = \binom{n+2-1}{2-1} = n+1.$$
   1.3 $\binom{-1/2}{n} = \left(-\frac{1}{4}\right)^n \binom{2n}{n}$.
2.
   2.1 $G(x) = (1 + x + x^2 + x^3)^7$.
   2.2 $G(x) = \left(\frac{1}{1-x} - x^4\right)^7$.

Playing blackjack with dice

Suppose we acquire the following (slightly simpler) non-standard die:

If we keep rolling the die until the total is at least 21, what is the probability that we hit 21 exactly?

Playing blackjack with dice

Suppose we acquire the following (slightly simpler) non-standard die:

⚀ ⚀ ⚀ ⚀ ⚁ ⚁

If we keep rolling the die until the total is at least 21, what is the probability that we hit 21 exactly?

1. The outcomes of one roll are given by $D = \frac{2}{3} \cdot$ ⚀ $+ \frac{1}{3} \cdot$ ⚁.

Playing blackjack with dice

Suppose we acquire the following (slightly simpler) non-standard die:

⚀ ⚀ ⚀ ⚀ ⚁ ⚁

If we keep rolling the die until the total is at least 21, what is the probability that we hit 21 exactly?

1. The outcomes of one roll are given by $D = \frac{2}{3} \cdot \text{⚀} + \frac{1}{3} \cdot \text{⚁}$.
2. Two rolls are described by $D^2 = \frac{4}{9} \cdot \text{⚀}^2 + \frac{4}{9} \cdot \text{⚀⚁} + \frac{1}{9} \cdot \text{⚁}^2$.

Playing blackjack with dice

Suppose we acquire the following (slightly simpler) non-standard die:

⚀ ⚀ ⚀ ⚀ ⚁ ⚁

If we keep rolling the die until the total is at least 21, what is the probability that we hit 21 exactly?

1. The outcomes of one roll are given by $D = \frac{2}{3} \cdot \text{⚀} + \frac{1}{3} \cdot \text{⚁}$.
2. Two rolls are described by $D^2 = \frac{4}{9} \cdot \text{⚀}^2 + \frac{4}{9} \cdot \text{⚀⚁} + \frac{1}{9} \cdot \text{⚁}^2$.
3. For any number of rolls, take $D^0 + D^1 + D^2 + D^3 + \cdots$. This is a geometric series, which sums to $\frac{1}{1 - \frac{2}{3} \cdot \text{⚀} - \frac{1}{3} \cdot \text{⚁}}$.

Playing blackjack with dice

Suppose we acquire the following (slightly simpler) non-standard die:

⚀ ⚀ ⚀ ⚀ ⚁ ⚁

If we keep rolling the die until the total is at least 21, what is the probability that we hit 21 exactly?

1. The outcomes of one roll are given by $D = \frac{2}{3} \cdot \text{⚀} + \frac{1}{3} \cdot \text{⚁}$.
2. Two rolls are described by $D^2 = \frac{4}{9} \cdot \text{⚀}^2 + \frac{4}{9} \cdot \text{⚀}\text{⚁} + \frac{1}{9} \cdot \text{⚁}^2$.
3. For any number of rolls, take $D^0 + D^1 + D^2 + D^3 + \cdots$. This is a geometric series, which sums to $\frac{1}{1 - \frac{2}{3} \cdot \text{⚀} - \frac{1}{3} \cdot \text{⚁}}$.
4. To find the total of each roll, let $\text{⚀} = x$ and $\text{⚁} = x^2$:

$$P(x) = \frac{1}{1 - \frac{2}{3}x - \frac{1}{3}x^2}.$$

What can we do with this thing?

Approach #0: Extract the first few terms

Given the expression $P(x) = \frac{1}{1 - \frac{2}{3}x - \frac{1}{3}x^2}$, what can we do? We would like to know the coefficient of x^{21} for this particular problem; in general, we want to know a formula for the coefficient of x^n.

What can we do with this thing?

Approach #0: Extract the first few terms

Given the expression $P(x) = \frac{1}{1-\frac{2}{3}x-\frac{1}{3}x^2}$, what can we do? We would like to know the coefficient of x^{21} for this particular problem; in general, we want to know a formula for the coefficient of x^n.

Think of $P(x)$ as $p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots$. We can obtain p_0 easily: $p_0 = P(0) = 1$.

What can we do with this thing?

Approach #0: Extract the first few terms

Given the expression $P(x) = \frac{1}{1-\frac{2}{3}x-\frac{1}{3}x^2}$, what can we do? We would like to know the coefficient of x^{21} for this particular problem; in general, we want to know a formula for the coefficient of x^n.

Think of $P(x)$ as $p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots$. We can obtain p_0 easily: $p_0 = P(0) = 1$.

Now that we know $P(x) = 1 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots$, we can manipulate this to bring p_1 to the front. We have

$$\frac{P(x) - 1}{x} = p_1 + p_2 x + p_3 x^2 + \cdots$$

but we can also work out $\frac{P(x)-1}{x} = \frac{\frac{2}{3}+\frac{1}{3}x}{1-\frac{2}{3}x-\frac{1}{3}x^2}$. Evaluating this at 0 gives us $p_1 = \frac{2}{3}$.

What can we do with this thing?

Approach #0: Extract the first few terms

Given the expression $P(x) = \frac{1}{1-\frac{2}{3}x-\frac{1}{3}x^2}$, what can we do? We would like to know the coefficient of x^{21} for this particular problem; in general, we want to know a formula for the coefficient of x^n.

Think of $P(x)$ as $p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots$. We can obtain p_0 easily: $p_0 = P(0) = 1$.

Now that we know $P(x) = 1 + p_1x + p_2x^2 + p_3x^3 + \cdots$, we can manipulate this to bring p_1 to the front. We have

$$\frac{P(x)-1}{x} = p_1 + p_2x + p_3x^2 + \cdots$$

but we can also work out $\frac{P(x)-1}{x} = \frac{\frac{2}{3}+\frac{1}{3}x}{1-\frac{2}{3}x-\frac{1}{3}x^2}$. Evaluating this at 0 gives us $p_1 = \frac{2}{3}$. We could keep going, but 21 is a long way away.

What can we do with this thing?

Approach #1: Find a recursive formula

We can also rewrite $P(x) = \frac{1}{1-\frac{2}{3}x-\frac{1}{3}x^2}$ as

$$\left(1 - \frac{2}{3}x - \frac{1}{3}x^2\right) P(x) = 1 \quad \Leftrightarrow \quad P(x) = 1+\frac{2}{3}x\, P(x)+\frac{1}{3}x^2\, P(x).$$

What can we do with this thing?

Approach #1: Find a recursive formula

We can also rewrite $P(x) = \frac{1}{1-\frac{2}{3}x-\frac{1}{3}x^2}$ as

$$\left(1 - \frac{2}{3}x - \frac{1}{3}x^2\right) P(x) = 1 \quad \Leftrightarrow \quad P(x) = 1+\frac{2}{3}x\, P(x)+\frac{1}{3}x^2\, P(x).$$

If we take the coefficient of x^n, for $n \geq 1$, we get

$$\begin{aligned} [x^n]P(x) &= [x^n]\left(1 + \frac{2}{3}x\, P(x) + \frac{1}{3}x^2\, P(x)\right) \\ &= [x^n]\left(\frac{2}{3}x\, P(x)\right) + [x^n]\left(\frac{1}{3}x^2\, P(x)\right) \\ &= \frac{2}{3}[x^{n-1}]P(x) + \frac{1}{3}[x^{n-2}]P(x). \end{aligned}$$

What can we do with this thing?

Approach #1: Find a recursive formula

We can also rewrite $P(x) = \frac{1}{1-\frac{2}{3}x-\frac{1}{3}x^2}$ as

$$\left(1 - \frac{2}{3}x - \frac{1}{3}x^2\right) P(x) = 1 \quad \Leftrightarrow \quad P(x) = 1 + \frac{2}{3}x\,P(x) + \frac{1}{3}x^2\,P(x).$$

If we take the coefficient of x^n, for $n \geq 1$, we get

$$\begin{aligned}[x^n]P(x) &= [x^n]\left(1 + \frac{2}{3}x\,P(x) + \frac{1}{3}x^2\,P(x)\right) \\ &= [x^n]\left(\frac{2}{3}x\,P(x)\right) + [x^n]\left(\frac{1}{3}x^2\,P(x)\right) \\ &= \frac{2}{3}[x^{n-1}]P(x) + \frac{1}{3}[x^{n-2}]P(x).\end{aligned}$$

This gives us a recursive formula: $p_n = \frac{2}{3}p_{n-1} + \frac{1}{3}p_{n-2}$.

What can we do with this thing?

Approach #2: Use partial fractions

Since $1 - \frac{2}{3}x - \frac{1}{3}x^2$ factors as $(1-x)(1+\frac{1}{3}x)$, we can write

$$P(x) = \frac{1}{1 - \frac{2}{3}x - \frac{1}{3}x^2} = \frac{A}{1-x} + \frac{B}{1+\frac{1}{3}x}.$$

What can we do with this thing?

Approach #2: Use partial fractions

Since $1 - \frac{2}{3}x - \frac{1}{3}x^2$ factors as $(1-x)(1+\frac{1}{3}x)$, we can write

$$P(x) = \frac{1}{1 - \frac{2}{3}x - \frac{1}{3}x^2} = \frac{A}{1-x} + \frac{B}{1+\frac{1}{3}x}.$$

To find A and B, just set x to some trial values:

$$\begin{cases} 1 = A + B & (x = 0) \\ \frac{1}{0} = \frac{1}{0}A + \frac{3}{4}B & (x = 1) \end{cases}$$

What can we do with this thing?

Approach #2: Use partial fractions

Since $1 - \frac{2}{3}x - \frac{1}{3}x^2$ factors as $(1-x)(1+\frac{1}{3}x)$, we can write

$$P(x) = \frac{1}{1 - \frac{2}{3}x - \frac{1}{3}x^2} = \frac{A}{1-x} + \frac{B}{1+\frac{1}{3}x}.$$

To find A and B, just set x to some trial values:

$$\begin{cases} 1 = A + B & (x = 0) \\ -\frac{1}{4} = -\frac{1}{2}A + \frac{1}{2}B & (x = 3) \end{cases}$$

What can we do with this thing?

Approach #2: Use partial fractions

Since $1 - \frac{2}{3}x - \frac{1}{3}x^2$ factors as $(1-x)(1+\frac{1}{3}x)$, we can write

$$P(x) = \frac{1}{1 - \frac{2}{3}x - \frac{1}{3}x^2} = \frac{A}{1-x} + \frac{B}{1+\frac{1}{3}x}.$$

To find A and B, just set x to some trial values:

$$\begin{cases} 1 = A + B & (x = 0) \\ -\frac{1}{4} = -\frac{1}{2}A + \frac{1}{2}B & (x = 3) \end{cases}$$

which we can solve to get

$$P(x) = \frac{3/4}{1-x} + \frac{1/4}{1+\frac{1}{3}x}$$

What can we do with this thing?

Approach #2: Use partial fractions

Since $1 - \frac{2}{3}x - \frac{1}{3}x^2$ factors as $(1-x)(1+\frac{1}{3}x)$, we can write

$$P(x) = \frac{1}{1 - \frac{2}{3}x - \frac{1}{3}x^2} = \frac{A}{1-x} + \frac{B}{1+\frac{1}{3}x}.$$

To find A and B, just set x to some trial values:

$$\begin{cases} 1 = A + B & (x = 0) \\ -\frac{1}{4} = -\frac{1}{2}A + \frac{1}{2}B & (x = 3) \end{cases}$$

which we can solve to get

$$P(x) = \frac{3/4}{1-x} + \frac{1/4}{1+\frac{1}{3}x} = \frac{3}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n x^n$$

What can we do with this thing?

Approach #2: Use partial fractions

Since $1 - \frac{2}{3}x - \frac{1}{3}x^2$ factors as $(1-x)(1+\frac{1}{3}x)$, we can write

$$P(x) = \frac{1}{1 - \frac{2}{3}x - \frac{1}{3}x^2} = \frac{A}{1-x} + \frac{B}{1+\frac{1}{3}x}.$$

To find A and B, just set x to some trial values:

$$\begin{cases} 1 = A + B & (x = 0) \\ -\frac{1}{4} = -\frac{1}{2}A + \frac{1}{2}B & (x = 3) \end{cases}$$

which we can solve to get

$$P(x) = \frac{3/4}{1-x} + \frac{1/4}{1+\frac{1}{3}x} = \frac{3}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n x^n$$

which means $p_n = \frac{3}{4} + \frac{1}{4}\cdot(-\frac{1}{3})^n$, and $p_{21} = \frac{3}{4} - \frac{1}{3^{21}}$.

Exercises

1. Let $J(x) = \frac{x}{(1+x)(1-2x)}$.
   1.1 Find a recursive formula for $J_n = [x^n]J(x)$.
   1.2 Write $J(x)$ as a sum of partial fractions and find the closed form of J_n.
2. Suppose you are flipping a fair coin over and over again.
   2.1 Write a formula for $G(x)$ such that $[x^n]G(x)$ is the probability it will take n flips to see an outcome of tails.
   2.2 What meaning does $G(x)^2$ have?
3. Let $F(x) = \frac{x}{1-x-x^2}$. Confirm that $[x^n]F(x)$ is the n-th Fibonacci number by:
   3.1 Checking that the first few terms $F_0, F_1, F_2, \ldots$ are correct, and
   3.2 Checking that the right recursive formula holds.

Solutions

1.
   1.1 $J_n = J_{n-1} + 2J_{n-2}$.

   1.2 $J(x) = \frac{1/3}{1-2x} - \frac{1/3}{1+x}$, and $J_n = \frac{2^n-(-1)^n}{3}$.

2.
   2.1 $G(x) = \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots = \frac{x}{2-x}$.

   2.2 $[x^n]G(x)^2$ is the probability it will take n flips to see tails twice.

3.
   3.1 We get $F_0 = F(0) = 0$. Shifting $F(x)$ over gives $\frac{F(x)-0}{x} = \frac{1}{1-x-x^2}$, which tells us $F_1 = 1$. Shifting $F(x)$ over again gives $\frac{F(x)-0-x}{x^2} = \frac{1+x}{1-x-x^2}$, which tells us $F_2 = 1$.

   3.2 We can rewrite $F(x) = \frac{x}{1-x-x^2}$ as $F(x) = 1 + xF(x) + x^2F(x)$. Taking the coefficient of x^n, for $n \geq 1$, gives us

   $$\begin{aligned}[x^n]F(x) &= [x^n](xF(x)) + [x^n](x^2F(x)) \\ &= [x^{n-1}]F(x) + [x^{n-2}]F(x)\end{aligned}$$

   so the recursive formula $F_n = F_{n-1} + F_{n-2}$ holds.