Generating Functions II

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Warm-up problems

1. Solve the recursion $a_{n+1} = 2a_n$, $a_0 = 1$ by using **common sense**.
2. Solve the recursion $b_{n+1} = 2b_n + 1$, $b_0 = 1$ by using common sense and **trickery**.
3. Solve the recursion $c_{n+1} = 2c_n + n$, $c_0 = 1$ by using common sense, trickery, and **persistence**.
4. Solve the recursion $d_{n+2} = d_n + d_{n+1} + 2^n$, $d_0 = d_1 = 1$ by using common sense, trickery, persistence, and **a familiarity with the Fibonacci numbers**.

Warm-up solutions

1. $a_n = 2^n$.
2. $b_n = 2^{n+1} - 1$.
3. $c_n = 2^{n+1} - n - 1$.
4. $d_n = 2^n - F_n$, where F_n is the n^{th} Fibonacci number (with $F_0 = 0$ and $F_1 = 1$).

Formal introduction to generating functions

If we have a sequence $a_0, a_1, a_2, a_3, \ldots$ that we like very much, we make a generating function for it by computing the sum

$$A(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Often there is some nice expression for what this sum is. For example, if the sequence is $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots,$ we get the generating function $(1+x)^n$.

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Reasons to use generating functions:

- We get short descriptions of complicated sequences.
- We can manipulate sequences in useful ways with simple algebra.

Recurrences

If you lack common sense, trickery, and persistence, you can use generating functions to solve recurrences. For example, take the recurrence $a_{n+1} = 2a_n + 1$ with $a_0 = 1$.

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$$\sum_{n=0}^{\infty} a_{n+1} x^n = \sum_{n=0}^{\infty} 2a_n x^n + \sum_{n=0}^{\infty} x^n$$

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\frac{A(x) - a_0}{x} &= 2A(x) + \frac{1}{1-x}
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This splits up as $A(x) = \frac{2}{1-2x} - \frac{1}{1-x}$, so $a_n = 2 \cdot 2^n - 1$.

Adding up Fibonacci numbers

The generating function for the Fibonacci numbers is $F(x) = \frac{x}{1-x-x^2}$. Suppose we want to compute $S_n = F_0 + F_1 + F_2 + \cdots + F_n$. How can we do this using generating functions?

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Let $S(x) = \sum_{n=0}^{\infty} S_n x^n$. Then:

$$
\begin{aligned}
S(x) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} F_k\right)x^n = \sum_{0\le k\le n} F_k x^n \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} F_k x^n = \sum_{k=0}^{\infty}\frac{F_k x^k}{1-x} \\
&= \frac{1}{1-x}\sum_{k=0}^{\infty} F_k x^k = \frac{1}{1-x}F(x) = \frac{x}{(1-x)(1-x-x^2)}.
\end{aligned}
$$

Adding up Fibonacci numbers

We can use "partial partial fractions" on $S(x) = \frac{x}{(1-x)(1-x-x^2)}$.
Write:

$$\frac{x}{(1-x)(1-x-x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1-x-x^2}.$$

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By setting x to a few trial values, we get:

$$\begin{cases} 0 = A + C & (x = 0) \\ -\frac{1}{2} = \frac{1}{2}A - B + C & (x = -1) \\ 4 = 2A + 2B + 4C & (x = \frac{1}{2}) \end{cases}$$

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We get $A = -1$, $B = C = 1$, so

$$S(x) = \frac{x+1}{1-x-x^2} - \frac{1}{1-x} = F(x) + \frac{F(x)}{x} - \frac{1}{1-x}.$$

Therefore $S_n = F_n + F_{n+1} - 1 = F_{n+2} - 1$.

Convolutions, again

There is another reason why $\frac{F(x)}{1-x}$ gave us the partial sums. We can write this as:

$$(F_0 + F_1x + F_2x^2 + F_3x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots).$$

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$$(F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots).$$

When we multiply this out, we get the following x^n terms:

$$(F_n x^n) \cdot 1 + (F_{n-1} x^{n-1}) \cdot x + \cdots + (F_1 x) \cdot x^{n-1} + F_0 \cdot x^n.$$

The total coefficient of x^n is $F_n + F_{n-1} + \cdots + F_1 + F_0$.

Exercises

1. Find the coefficient of x^{100} in $\frac{(1+x)^{10}}{1-x}$.
2. Solve the recurrence $a_n = a_0 + a_1 + \cdots + a_{n-1}$, with $a_0 = 1$.
3. Compute the sum $F_0 + F_2 + F_4 + \cdots + F_{2n-2} + F_{2n}$.

Exercises

1. The coefficient of x^{100} (and any x^n for $n \geq 10$) is $2^{10} = 1024$.
2. $a_n = 2^{n-1}$ for $n \geq 1$. The generating function (if you took that approach) is

$$A(x) = \frac{1-x}{1-2x} = \frac{1/2}{1-2x} + \frac{1}{2}.$$

3. $F_0 + F_2 + F_4 + \cdots + F_{2n-2} + F_{2n} = F_{2n+1} - 1$. In generating function language:

$$\frac{F(x)}{1-x^2} = \frac{F(x)}{x} - \frac{1}{1-x^2}.$$

   This also implies that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

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2. How can we find $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n-(n \bmod 2)}$ using generating functions?

   Setting $x = 1$ converts $\binom{n}{k}x^k$ into just $\binom{n}{k}$ for all k. But setting $x = -1$ converts $\binom{n}{k}x^k$ into $\binom{n}{k}$ for even k, and $-\binom{n}{k}$ for all k.

   Therefore $\frac{f(1)+f(-1)}{2}$ will preserve the even terms and cancel the odd terms, giving us $\frac{(1+1)^n+(1-1)^n}{2} = 2^{n-1}$.

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We can solve this in a similar way. Let $\omega = e^{2\pi i/5}$: a complex number satisfying $\omega^5 = 1$. Since $x^5 - 1$ factors as $(x-1)(x^4 + x^3 + x^2 + x + 1)$, and $\omega \neq 1$, we also have $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

Let $f(x)$ be a generating function. What is $\frac{f(1)+f(\omega)+\cdots+f(\omega^4)}{5}$?

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Let $f(x)$ be a generating function. What is $\frac{f(1)+f(\omega)+\cdots+f(\omega^4)}{5}$?

Each x^n becomes $\frac{1+\omega^n+\omega^{2n}+\omega^{3n}+\omega^{4n}}{5}$. If n is a multiple of 5, this becomes $\frac{1+1+1+1+1}{5} = 1$. Otherwise, the numerator is a permutation of $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

If we do this for the probability generating function of rolling 10 dice, we get the answer we want.

Another dice problem: Solution

The probability generating function of one die roll is $\frac{x+x^2+x^3+x^4+x^5+x^6}{6}$. So the p.g.f. for rolling 10 dice is

$$f(x) = \left(\frac{x + x^2 + x^3 + x^4 + x^5 + x^6}{6}\right)^{10}.$$

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Next, we compute $f(1)$, $f(\omega)$, $f(\omega^2)$, $f(\omega^3)$, and $f(\omega^4)$. The first is easy: $f(1) = 1$.

We can simplify $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6$ to just $\omega + \omega^2(1 + \omega + \omega^2 + \omega^3 + \omega^4) = \omega$. So $f(\omega) = \left(\frac{\omega}{6}\right)^{10} = \frac{1}{6^{10}}$. The same thing happens for ω^2, ω^3, and ω^4.

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Therefore the probability we want is

$$\frac{1 + \frac{1}{6^{10}} + \frac{1}{6^{10}} + \frac{1}{6^{10}} + \frac{1}{6^{10}}}{5} = \frac{1}{5} + \frac{4}{5 \cdot 6^{10}}.$$

Exercises

1. Compute $\binom{99}{0} + \binom{99}{3} + \binom{99}{6} + \cdots + \binom{99}{99}$.
2. Compute $\binom{99}{1} + \binom{99}{4} + \binom{99}{7} + \cdots + \binom{99}{97}$.
3. Using the g.f. $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$, find the sum

$$\sum_{n=0}^{\infty} \frac{n}{2^n}.$$

Solutions

1. Let $f(x) = (1+x)^{99}$. Then we take $\frac{f(1)+f(e^{2\pi i/3})+f(e^{4\pi i/3})}{3} = \frac{2^{99}-2}{3}$.
2. Here, we do the same thing to $f(x) = x^2(1+x)^{99}$, and get $\frac{2^{99}+1}{3}$.
3. Taking $\frac{1}{(1-\frac{1}{2})^2}$ gives us 4, but this is actually the sum $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$. So we subtract off $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$, and get a total of 2.

Bonus problem

Let T_n be the number of triangles you can make with integer sides and perimeter n.

Find the generating function $T(x) = \sum_{n=0}^{\infty} T_n x^n$.

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Find the generating function $T(x) = \sum_{n=0}^{\infty} T_n x^n$.

This amounts to choosing a, b, c such that $a \leq b \leq c$ and $c < a + b$. But that is equivalent to choosing $p = b - a$, $q = c - b$, and $r = a + b - c$ such that $p \geq 0$, $q \geq 0$, and $r > 0$. This yields the generating function

$$T(u, v, w) = \frac{1}{1-u} \cdot \frac{1}{1-v} \cdot \frac{w}{1-w}.$$

Given p, q, r, the perimeter is $2p + 4q + 3r = 2(b-a) + 4(c-b) + 3(a+b-c) = a + b + c$. So we substitute $u = x^2$, $v = x^4$, $w = x^3$ to get

$$T(x) = \frac{x^3}{(1-x^2)(1-x^3)(1-x^4)}.$$