Proportionality and Area Chasing

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JV Practice (Proportionality)

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1 Intercept Theorem

1.1 The theorem

Let two non parallel l_1, l_2 intersect at point X. Let two lines intersect l_1 at A, B and l_2 at C, D in this order. Then these two lines are parallel if and only if $\frac{XA}{XB} = \frac{XC}{XD}$.

1.2

Let $ABCD$ be a trapezoid and $AC \cap BD = \{O\}$. If $AC = 20$ and $\frac{DO}{BO} = \frac{3}{7}$, compute the lengths of AO and OC.

1.3

Let l_1, l_2, l_3 be three lines that meet at O. Let $A, A' \in l_1, B, B' \in l_2, C, C' \in l_3$ such that $AB \parallel A'B'$ and $BC \parallel B'C'$. Prove that $AC \parallel A'C'$.

1.4

In $\triangle\ ABC$ and $E, H \in (AB), F \in AC, G \in BC$ such that $EF \parallel BC, FG \parallel AB, GH \parallel AC$. Prove $AE \cdot AH = BE \cdot BH$.

1.5

Let $ABCD$ be a rectangle. Take $E \in (AB), G \in (AD$ and construct F such that $ABCD$ and rectangle $AEFG$ have equal area. Prove that $DE \parallel BG$.

2 The Angle Bisector Theorem

2.1 The theorem

Consider $\triangle ABC$. Let $D \in (BC)$. We have that AD is the angle bisector of $BAC \iff \frac{AB}{AC} = \frac{BD}{DC}$. Prove the generalization: if $D \in BC$, then $\frac{BD}{DC} = \frac{AB}{AC}\frac{\sin(DAB)}{\sin(DAC)}$. Furthermore, if AD is the exterior bisector of $BAC, D \in BC$, then $\frac{AB}{AC} = \frac{BD}{CD}$.

2.2

Consider the rectangle $ABCD$. The angle bisectors of BAC and DAC intersect lines BC and CD in M, N respectively. Prove that $\frac{MB}{MC} + \frac{ND}{NC}>1$.

2.3

It's bash time! Prove that if AD is the bisector, $a = BC, b = AC, c = BC$, then $AD = \frac{2bc}{b+c}\cos(\frac{A}{2})$.

2.4

Let $\triangle ABC$ be a triangle with $AB = 20, BC = 30$, and let BD be the angle bisector of $\angle ABC, D \in (AC)$. If $E \in (BC), F \in AC$ such that $DE \parallel AB, EF \parallel BD$ and $AD - CF = 1$, find the length of AC.

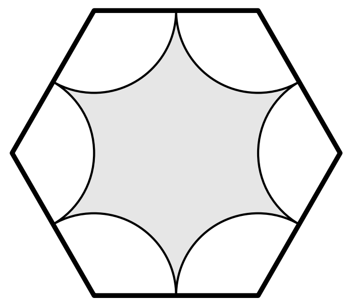
2.5

Let AM be a median in $\triangle ABC$. The bisector of AMB intersects AB in M, while the bisector of AMC intersects AC in P. Prove that $NP \parallel BC$.

Varsity Practice (Area Chasing)

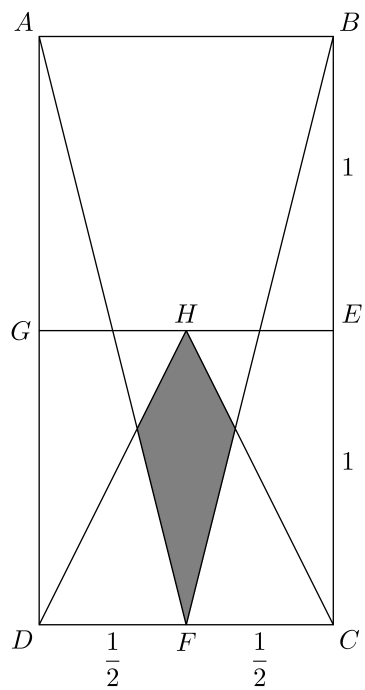
Gunmay Handa

1. (2014 AMC 10A) A regular hexagon has side length 6. Congruent arcs with radius 3 are drawn with the center at each of the vertices, creating circular sectors as shown. The region inside the hexagon but outside the sectors is shaded as shown What is the area of the shaded region?



2. (CMIMC 2016) Let $ABCD$ be an isosceles trapezoid with $AD = BC = 15$ such that the distance between its bases AB and CD is 7. Suppose further that the circles with diameters $\overline{AD}$ and $\overline{BC}$ are tangent to each other. What is the area of the trapezoid?

3. (2014 AMC 10A) In rectangle $ABCD$, $AB = 1$, $BC = 2$, and points E, F, and G are midpoints of $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point H is the midpoint of $\overline{GE}$. What is the area of the shaded region?



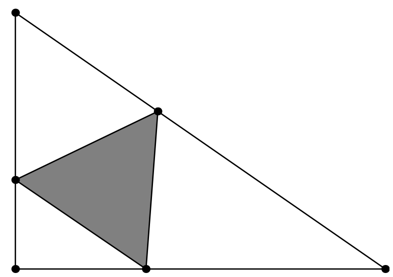
4. (CMIMC 2019) Let $MATH$ be a trapezoid with $MA = AT = TH = 5$ and $MH = 11$. Point S is the orthocenter of $\triangle ATH$. Compute the area of quadrilateral $MASH$.

5. (CMIMC 2018) Let ABC be a triangle with side lengths 5, $4\sqrt{2}$, and 7. What is the area of the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$?

6. (NIMO 12.2) Let ABC be an equilateral triangle. Denote by D the midpoint of $\overline{BC}$, and denote the circle with diameter $\overline{AD}$ by Ω. If the region inside Ω and outside $\triangle ABC$ has area $800\pi - 600\sqrt{3}$, find the area of $\triangle ABC$.

7. (BMT 2017) Let ABC be a triangle with $AB = 3$, $BC = 5$, $AC = 7$, and let P be a point in its interior. If G_A, G_B, G_C are the centroids of $\triangle PBC, \triangle PAC, \triangle PAB$, respectively, find the maximum possible area of $\triangle G_AG_BG_C$.

8. (2017 AIME I)* What is the area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, and $\sqrt{37}$, as shown?



9. (CMIMC 2019)* Suppose ABC is a triangle, and define B_1 and C_1 such that $\triangle AB_1C$ and $\triangle AC_1B$ are isosceles right triangles on the exterior of $\triangle ABC$ with right angles at B_1 and C_1, respectively. Let M be the midpoint of $\overline{B_1C_1}$; if $B_1C_1 = 12$, $BM = 7$ and $CM = 11$, what is the area of $\triangle ABC$?

10. (2012 AIME I)* Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. What is the largest possible area of the triangle?