Circles

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1 Lecture

There are two main aspects regarding circles: angle relationships and length relationships. We explore both of them here.

2 Angles

Here are a few facts that are worth knowing. These are provable by simple angle chasing.

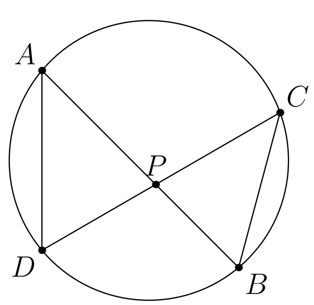
- If points A, B, and C are placed on a circle, then $\angle ABC = \frac{1}{2}\widehat{AC}$.
- Suppose points A, B, C, and D are placed on a circle in this order. Let $E = AC \cap BD$. Then
$$\angle AED = \frac{\widehat{AD} + \widehat{BC}}{2}.$$
- In the same configuration as before, if $F = AB \cap CD$, then
$$\angle AFD = \frac{\widehat{AD} - \widehat{BC}}{2}.$$
As a particular consequence, what happens if $B = C$?

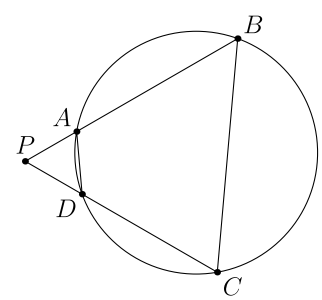
3 Power of a Point

The following theorem is one very well known to those having experienced geometry; the theorem of the power of a point:

Theorem. Let O be a circle, and a point P on the same plane. Let line l_1 through P interect O at A and B, and let line l_2 through P intersect O through C and D. Then we have that

$$PA \cdot PB = PC \cdot PD.$$

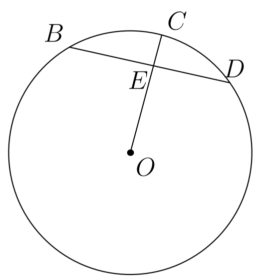




Remark: Note that there are multiple configurations which are possible, but that the theorem still holds.

Hint: To prove the above fact, can you find similar triangles?

Example. DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the center of the circle. Extend OE to the circle such that the ray OE intersects the circle at C. If $EC = 1$, what is the radius of the circle? (Canada 1971)



4 Radical Axis

Note that the above theorem essentially states the following: if a point P is fixed, and line ℓ intersects a fixed circle Ω at two points A and B, then the quantity $PA \cdot PB$ is fixed regardless of the choice of ℓ. This allows us to define the *power* of P with respect to Ω, $P_\Omega(P)$, to be equal to this fixed quantity. It is not hard to show that this equals $OP^2 - R_\Omega^2$, where O is the center of Ω and R_Ω is its radius.
We define the **radical axis** of two circles Γ_1 and Γ_2 to be the locus of all points P such that the power of P with respect to both circles are equal, or that

$$P_{\Gamma_1}(P) = P_{\Gamma_2}(P).$$

Alternatively, we can write this as the locus of points P such that

$$PO_1^2 - R_1^2 = PO_2^2 - R_2^2,$$

where R_1 and R_2 are the radii of the circles and O_1 and O_2 are the centers of the circles. Seem familiar? With this we can state a very useful theorem:

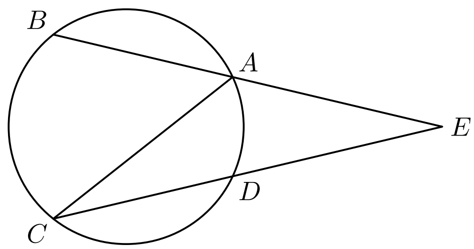
Theorem. Let Γ_1 and Γ_2 be two distinct circles with distinct centers O_1 and O_2. Then the radical axis of Γ_1 and Γ_2 is a line perpendicular to O_1O_2.

5 Problems

1. [AHSME 1977] In the figure, $\angle E = 40°$, and $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CD}$ have the same length. What is $\angle ACD$?

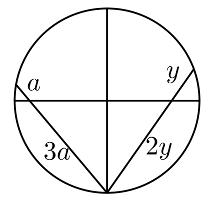
2. [AMC 10B 2008] Points A and B are on a circle of radius 5 and $AB = 6$. Point C is the midpoint of the minor arc AB. What is the length of the line segment AC?

3. [CMIMC 2016] Point A lies on the circumference of a circle Ω with radius 78. Point B is placed such that AB is tangent to the circle and $AB = 65$, while point C is located on Ω such that $BC = 25$. Compute the length of $\overline{AC}$.

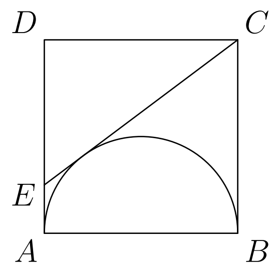


4. [AMC 10A 2013] In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

5. [Math League HS 2011-2012] In the diagram shown, a circle is divided by perpendicular diameters. One chord is divided into two parts in the ratio 2:1, and the other into two parts in the ratio 3:1. What is the ratio, larger to smaller, of the lengths of the two chords?

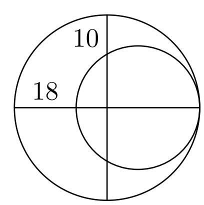


6. [AMC 10A 2004] Square $ABCD$ has side length 2. A semicircle with diameter AB is constructed inside the square, and the tangent to the semicircle from C intersects side AD at E. What is the length of CE?



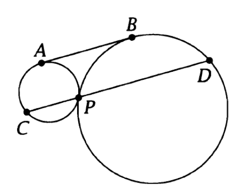
7. [HMMT 2009] Circle Ω has radius 13. Circle ω has radius 14 and its center P lies on the boundary of circle Ω. Points A and B lie on Ω such that chord AB has length 24 and is tangent to ω at point T. Find $AT \cdot BT$.

8. [Math League HS 2002-2003] Two perpendicular diameters are drawn in a circle. Another circle, tangent to the first at an endpoint of one of its diameters, cuts off segments of lengths 10 and 18 from the diameters, as in the diagram (which is not drawn to scale). How long is a diameter of the larger circle?



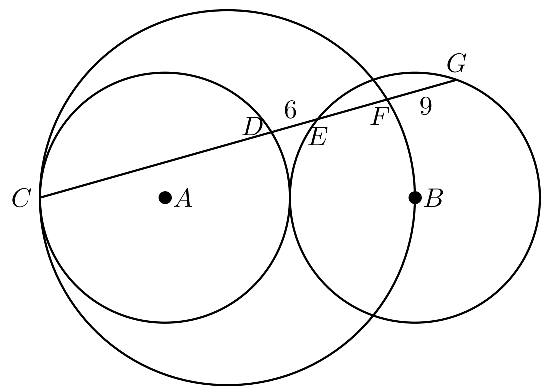
9. [AMC 12A 2012] Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

10. [ARML 1989] Two circles are externally tangent at point P, as shown. Segment $\overline{CPD}$ is parallel to the common external tangent $\overline{AB}$. If the radii of the circles are 2 and 18, compute the distance between the midpoints of $\overline{AB}$ and $\overline{CD}$.



11. [CMIMC 2016] Let ABC be a triangle with incenter I and incircle ω. It is given that there exist points X and Y on the circumference of ω such that $\angle BXC = \angle BYC = 90^\circ$. Suppose further that X, I, and Y are collinear. If $AB = 80$ and $AC = 97$, compute the length of BC.

12. [Math Prize for Girls 2015] In the diagram below, the circle with center A is congruent to and tangent to the circle with center B. A third circle is tangent to the circle with center A at point C and passes through point B. Points C, A, and B are collinear. The line segment $\overline{CDEFG}$ intersects the circles at the indicated points. Suppose that $DE = 6$ and $FG = 9$. Find AG.



13. In this problem, we prove that the radical axis is indeed a straight line.

    (a) Let A, B, C, and D be points in the plane. Show that $AC \perp BD$ if and only if $AB^2 + CD^2 = AD^2 + BC^2$.

    (b) Let Ω_1 and Ω_2 be two circles in the plane (which do not necessarily intersect). Find two points which necessarily have equal power with respect to both circles by considering the two external tangents to both Ω_1 and Ω_2.

    (c) Denote by X and Y the two points in the previous question. Let P be an arbitrary point in the plane. Show that $\mathcal{P}(P, \Omega_1) = \mathcal{P}(P, \Omega_2)$ if and only if P, X, and Y are collinear. Deduce the requested result.

14. [AIME 2005] Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.

15. Let $\triangle ABC$ be a triangle. Suppose that O and I_A are the circumcenter and A-excenter of $\triangle ABC$, and further suppose that E and F are the feet of the internal angle bisectors from $\angle B$ and $\angle C$ respectively. Show that $OI_A \perp EF$.

16. [ISL 2008] Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.