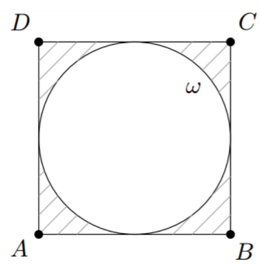
Area (Chasing)

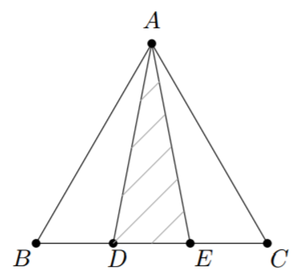
The last of the three major computational ideas in geometry is area. We touched on this a bit last lecture, but today we will explore it fully. There are, after all, several nontrivial things that can be done with area that make it quite a valuable tool.

1 Problems

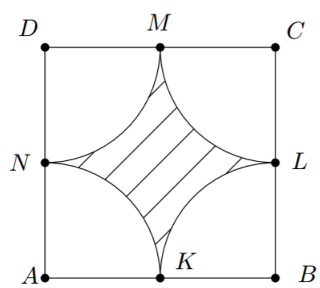
1. Compute the shaded areas below.



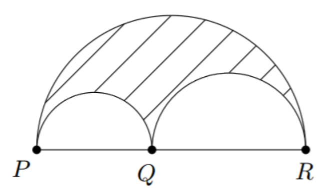
$ABCD$ square with $AB = 1$
ω incircle



ABC equilateral, $AB = 3$
$BD = DE = EC$

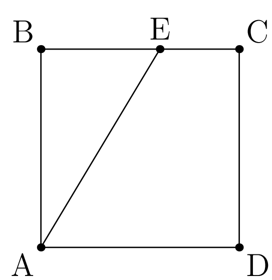


$ABCD$ square with $AB = 2$
K, L, M, N midpoints of sides
A, B, C, D centers of the arcs



PQ, QR, PR are diameters
$PQ = 3$ $\quad$ $QR = 4$

2. [AMC 10A 2013] Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE?



3. Let $\triangle ABC$ be an acute triangle, and let E and F be the feet of the perpendiculars from B to AC and from C to AB, respectively. Suppose $AB = 12$, $AC = 18$, and $BE = 7$. What is CF?

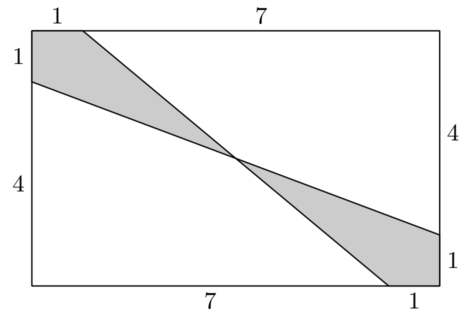
4. [AMC 10A 2011] A rectangular region is bounded by the graphs of the equations $y = a, y = -b, x = -c$, and $x = d$, where a, b, c, and d are all positive numbers. Which of the following represents the area of this region?

(A) $ac + ad + bc + bd$ **(B)** $ac - ad + bc - bd$ **(C)** $ac + ad - bc - bd$
(D) $-ac - ad + bc + bd$ **(E)** $ac - ad - bc + bd$

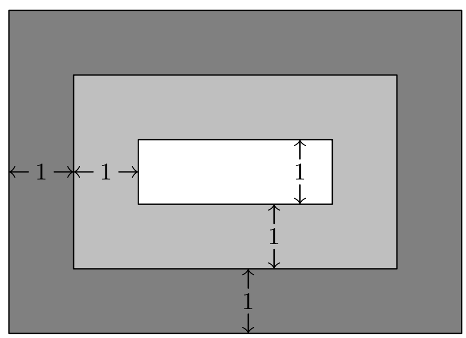
5. [NIMO 28, Own[1]] Trapezoid $ABCD$ is an isosceles trapezoid with $AD = BC$. Point P is the intersection of diagonals AC and BD. If the area of $\triangle ABP$ is 50 and the area of $\triangle CDP$ is 72, what is the area of the trapezoid?

   Hint: Compute the area of $\triangle BCP$. To do this, first find the ratio $AP : PC$, and use this to compare the areas of $\triangle ABP$ and $\triangle BCP$.

6. [AMC 10A 2016] What is the area of the shaded region of the given 8×5 rectangle?



7. [AMC 10B 2016] A rug is made with three different colors as shown. The areas of the three differently colored regions form an arithmetic progression. The inner rectangle is one foot wide, and each of the two shaded regions is 1 foot wide on all four sides. What is the length in feet of the inner rectangle?



8. Regular hexagon $ABCDEF$ is given in the plane. If the area of the triangle whose vertices are the midpoints of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ is 225, what is the area of $ABCDEF$?

9. [AMC 10A 2017] Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle as points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

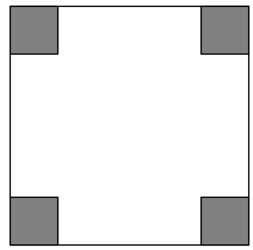
10. [Mandelbrot 2006-2007] Suppose that $ABCD$ is a trapezoid in which $\overline{AD} \parallel \overline{BC}$. Given $\overline{AC} \perp \overline{CD}$, $\overline{AC}$ bisects angle $\angle BAD$, and $area(ABCD) = 42$, then compute $area(ACD)$.

11. [AMC8 2015] One-inch squares are cut from the corners of this 5 inch square (see next page). What is the area in square inches of the largest square that can be fitted into the remaining space?

12. [Own] A rhombus has height 15 and diagonals with lengths in a ratio of $3 : 4$. What is the side length of the rhombus?

13. [AMC 10A 2013] Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

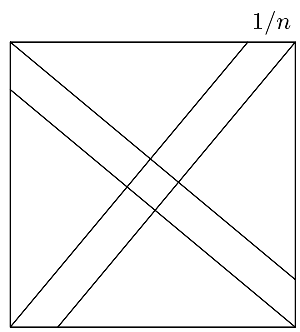
[1] but also kind of classical



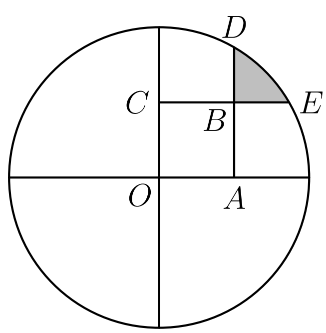
14. [CMIMC 2016, Own] Let ABC be a triangle. The angle bisector of $\angle B$ intersects AC at a point P, while the angle bisector of $\angle C$ intersects AB at a point Q. Suppose the area of $\triangle ABP$ is 27, the area of $\triangle ACQ$ is 32, and the area of $\triangle ABC$ is 72. What is BC?

15. Suppose $\triangle ABC$ satisfies $AB = 9$, $BC = 12$, and $CA = 15$. Let G denote the centroid of $\triangle ABC$. Points D, E, and F are the projections of G onto AB, BC, and CA respectively. Compute the area of $\triangle DEF$.

16. [AIME 1985] A small square is constructed inside a square of area 1 by dividing each side of the unit square into n equal parts, and then connecting the vertices to the division points closest to the opposite vertices. Find the value of n if the the area of the small square is exactly $1/1985$.

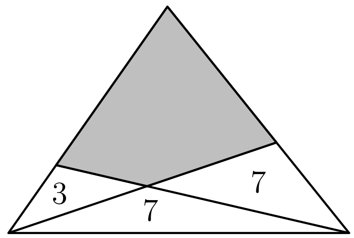


17. [AMC 10B 2006] A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides $\overline{AB}$ and $\overline{CB}$ are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by $\overline{BD}$, $\overline{BE}$, and the minor arc connecting D and E?

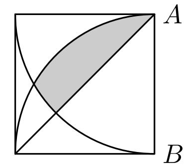


18. [AMC 10B 2006] A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7, as shown (see next page). What is the area of the shaded quadrilateral?

★ 19. [AMC 10A 2013] A unit square is rotated 45° about its center. What is the area of the region swept out by the interior of the square?



★ 20. [Mandelbrot 1992-1993] In the diagram, the curved paths are arcs of circles centered at vertices A and B of a square of side 6. Find the area of the shaded section.



★ 21. [Equation of a Line in Barycentric Coordinates] Let ABC be a triangle, and suppose ℓ is a line passing through its interior. Show that there exist real numbers u, v, and w, not all equaling zero, such that

$$u[BPC] + v[CPA] + w[APB] = 0$$

for all points $P \in \ell$ inside $\triangle ABC$.