Trigonometric Computation

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0.1 Lecture

- There are many different formulas for the area of a triangle. The ones that you'll probably need to know the most are summarized below:
$$A = \frac{1}{2}bh = \frac{1}{2}ab\sin C = rs = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)} = r_a(s-a).$$
Make sure you know how to derive these!

- Law of Sines: In any triangle ABC, we have
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

- Law of Cosines: In any triangle ABC, we have
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

- Ratio Lemma: In any triangle ABC, if D is a point on BC, then
$$\frac{BD}{DC} = \frac{BA\sin\angle BAD}{AC\sin\angle DAC}.$$
Note that this is a generalization of the Angle Bisector Theorem. Note further that this does not necessarily require that D lie on the segment $\overline{BC}$.

- Make sure you know your trig identities! Here are a few ones to keep in mind:
  - Negation: We have
  $$\sin(-\alpha) = -\sin\alpha \qquad \text{and} \qquad \cos(-\alpha) = \cos\alpha$$
  for all angles α.
  - Pythagorean Identity: For any angle α,
  $$\sin^2\alpha + \cos^2\alpha = 1.$$
  Be sure to memorize this one!
  - Sum and Difference Identities: For any angles α and β,
  $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha,$$
  $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$
  - Double Angle Identities: We have
  $$\sin(2A) = 2\sin A\cos A, \quad \cos(2A) = \cos^2 A - \sin^2 A, \quad \tan(2A) = \frac{2\tan A}{1-\tan^2 A}.$$
  - Half Angle Identities: We have
  $$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}, \quad \cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}, \quad \tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}.$$

0.2 Problems

1. Let $\triangle ABC$ satisfy $AB = 13$, $BC = 14$, $CA = 15$. Compute its area, inradius, and circumradius.

2. [AMC 12A 2012] A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin\theta$?

3. [AMC 12A 2003] An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

4. [NIMO 11] In triangle ABC, $\sin A \sin B \sin C = \frac{1}{1000}$ and $AB \cdot BC \cdot CA = 1000$. What is the area of triangle ABC?

5. [NIMO 12] Triangle ABC has sidelengths $AB = 14, BC = 15$, and $CA = 13$. We draw a circle with diameter AB such that it passes BC again at D and passes CA again at E. Find the circumradius of $\triangle CDE$.

6. Let $ABCD$ be a rectangle, and let P be a point inside $ABCD$ but not on either interior diagonal. Show that
$$\frac{\text{Area}(\triangle APC)}{\text{Area}(\triangle BPD)} = \frac{\tan\angle APC}{\tan\angle BPD}.$$

7. Let ABC be a triangle, and denote by H its orthocenter. Show that $AH = 2R\cos A$.

8. [AIME 2004] A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Compute the probability that the circle will not touch diagonal AC.

9. Prove the *Law of Tangents*: in any $\triangle ABC$, we have
$$\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}} = \frac{a+b}{a-b}.$$

10. Prove that a triangle $\triangle ABC$ is isosceles if and only if
$$a\cos B + b\cos C + c\cos A = \frac{a+b+c}{2}.$$

11. [NIMO 16] Let $\triangle ABC$ have $AB = 6$, $BC = 7$, and $CA = 8$, and denote by ω its circumcircle. Let N be a point on ω such that AN is a diameter of ω. Furthermore, let the tangent to ω at A intersect BC at T, and let the second intersection point of NT with ω be X. The length of $\overline{AX}$ can be written in the form $\frac{m}{\sqrt{n}}$ for positive integers m and n, where n is not divisible by the square of any prime. Find $100m + n$.

12. [Math Prize for Girls 2016] In the coordinate plane, consider points $A = (0, 0)$, $B = (11, 0)$, and $C = (18, 0)$. Line ℓ_A has slope 1 and passes through A. Line ℓ_B is vertical and passes through B. Line ℓ_C has slope -1 and passes through C. The three lines ℓ_A, ℓ_B, and ℓ_C begin rotating clockwise about points A, B, and C, respectively. They rotate at the same angular rate. At any given time, the three lines form a triangle. Determine the largest possible area of such a triangle.

13. [NIMO 8] The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.

14. [David Altizio] Let ABC be a triangle with $AB = 3$ and $AC = 4$. Points O and H are the circumcenter and orthocenter respectively of the triangle. if $OH \parallel BC$, then find $\cos A$.

15. Show that in $\triangle ABC$, we have

$$BC^3 \cos(B - C) + CA^3 \cos(C - A) + AB^3 \cos(A - B) = 3(BC)(CA)(AB).$$

16. [APMO 2000] Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and let O be the point in which the perpendicular at P to BA meets AN. Prove that $QO \perp BC$.