Geometry

Theorems about triangles

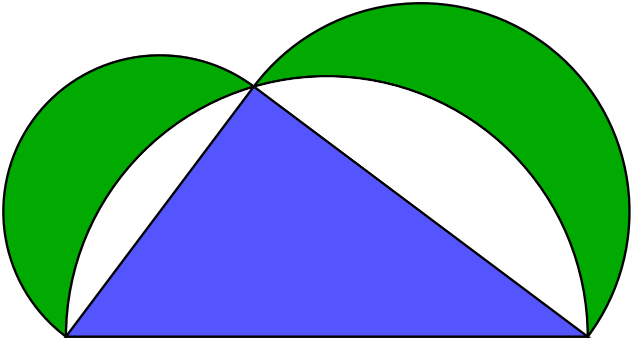
Misha Lavrov

ARML Practice 12/15/2013

Warm-up problem

Lunes of Hippocrates

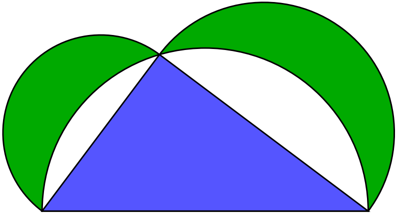
In the diagram below, the blue triangle is a right triangle with side lengths 3, 4, and 5.



What is the total area of the green shaded regions?

Solution

The lunes in the picture are formed by three semicircles whose diameters are the three sides of the triangle.



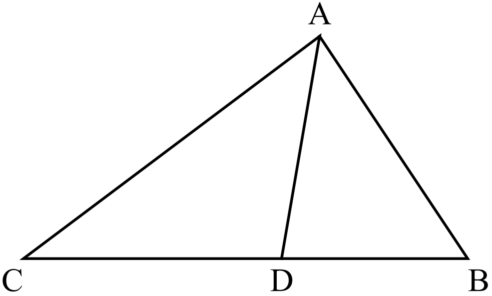
By the Pythagorean theorem, if we add the areas of the two small semicircles, and subtract the area of the larger semicircle, we get 0. (In this case, the areas are $\frac{9}{2}\pi$, 8π, and $\frac{25}{2}\pi$.)

But in the diagram, this is the difference between the green area and the blue area. So the green area is equal to the blue area, which we can compute to be 6.

The angle bisector theorem

Suppose that in the triangle $\triangle ABC$, AD is an angle bisector: $\angle BAD = \angle CAD$. Then

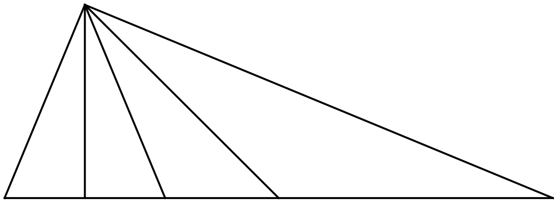
$$\frac{AB}{AC} = \frac{BD}{CD}.$$



I have three proofs of this theorem.

Angle bisector exercise

We are given a triangle with the following property: one of its angles is quadrisected (divided into four equal angles) by the height, the angle bisector, and the median from that vertex.



This property uniquely determines the triangle (up to scaling). Find the measure of the quadrisected angle.

(Hint: go wild with the angle bisector theorem.)

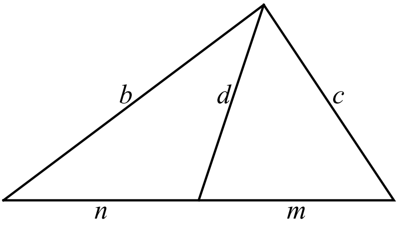
Solution

The base is partitioned into four segments in the ratio $x:x:y:2x+y$. Suppose the length of the left-hand side of the triangle is 1. Then the length of the angle bisector is also 1. Applying the angle bisector theorem to the large triangle, we see that the length of the right-hand side is $\frac{2x+2y}{2x} = 1+\frac{y}{x}$. But if we apply the angle bisector theorem to the left half of the triangle, we obtain $\frac{2x+y}{y} = 1+\frac{2x}{y}$ for the same length. Therefore $\frac{y}{x} = \frac{2x}{y}$, so $x:y = 1:\sqrt{2}$.

Now apply the angle bisector theorem a third time to the right triangle formed by the altitude and the median. The segments in the base are in the ratio $x:y = 1:\sqrt{2}$, so the altitude and the median form the same ratio. As this is a right triangle, it must be a 45°-45°-90° triangle. So the quadrisected angle is right.

Stewart's theorem

The line in the diagram below is no longer an angle bisector but just an arbitrary line. In addition to the labeled lengths, the base of the triangle has length $a = m + n$.



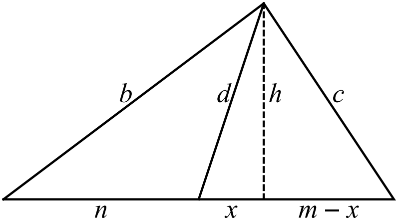
Stewart's Theorem. In this setting, $b^2m + c^2n = a(d^2 + mn)$.

It's conventional to memorize: "$man + dad = bmb + cnc$", or "A man and his dad put a bomb in the sink."

Stewart's theorem

Proof

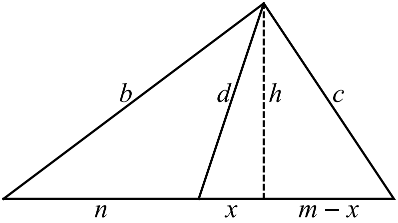
Draw the height, h, and label the unknown length in the base by x.



Stewart's theorem

Proof

Draw the height, h, and label the unknown length in the base by x.



Then we have:

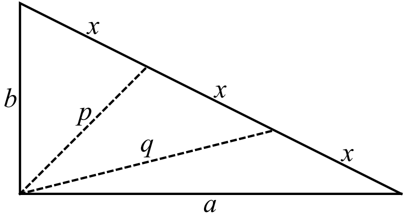
$$\begin{cases} b^2 = h^2 + (n+x)^2 \\ c^2 = h^2 + (m-x)^2 \\ d^2 = h^2 + x^2. \end{cases}$$

Eliminate first h and then x to obtain Stewart's theorem.

Problems

1. Two sides of a triangle are 4 and 9; the median drawn to the third side has length 6. Find the length of the third side.

2. A right triangle has legs a and b and hypotenuse c. Two segments from the right angle to the hypotenuse are drawn, dividing it into three equal parts of length $x = \frac{c}{3}$.



If the segments have length p and q, prove that $p^2 + q^2 = 5x^2$.

Solutions

1. If a is the length of the third side, then $m = n = \frac{a}{2}$, and we have

$$8a + \frac{81}{2}a = a\left(36 + \frac{1}{4}a^2\right)$$

which yields $a^2 = 50$ or $a = 5\sqrt{2}$.

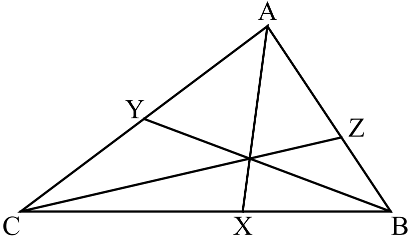
2. Applied first to p and then to q, Stewart's theorem yields two equations:

$$\begin{cases} a^2x + b^2(2x) = 3x(p^2 + 2x^2), \\ a^2(2x) + b^2x = 3x(q^2 + 2x^2). \end{cases}$$

Adding these, we get $(a^2 + b^2)(3x) = (p^2 + q^2 + 4x^2)(3x)$, so $p^2 + q^2 = a^2 + b^2 - 4x^2$. But $a^2 + b^2 = c^2 = 9x^2$, so $p^2 + q^2 = 5x^2$.

Ceva's theorem

In a triangle $\triangle ABC$, let X, Y, and Z be points on the sides opposite A, B, and C, respectively.



The AX, BY, and CZ meet at a single point if and only if:

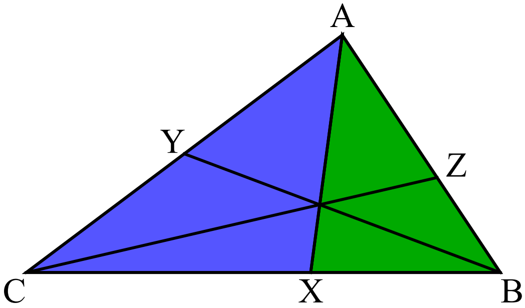
$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1.$$

(Terminology: we say AX, BY, CZ are *concurrent*.)

Ceva's theorem

Proof (one direction)

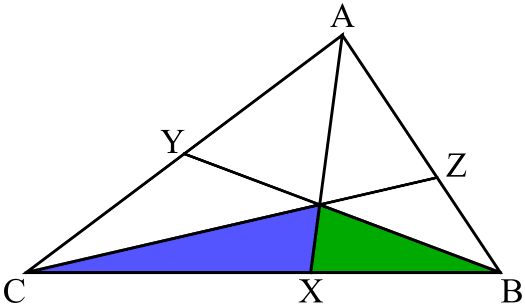
We have $\dfrac{BX}{XC} = \dfrac{[ABX]}{[AXC]}$.



Ceva's theorem

Proof (one direction)

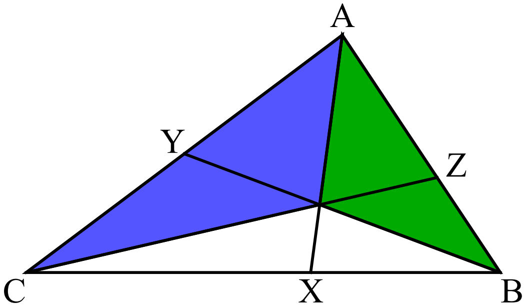
We have $\dfrac{BX}{XC} = \dfrac{[ABX]}{[AXC]} = \dfrac{[OBX]}{[OXC]}$.



Ceva's theorem

Proof (one direction)

We have $\dfrac{BX}{XC} = \dfrac{[ABX]}{[AXC]} = \dfrac{[OBX]}{[OXC]} = \dfrac{[ABO]}{[ACO]}$.



Therefore $\dfrac{AZ}{ZB} \cdot \dfrac{BX}{XC} \cdot \dfrac{CY}{YA} = \dfrac{[ACO]}{[BCO]} \cdot \dfrac{[ABO]}{[ACO]} \cdot \dfrac{[BCO]}{[ABO]} = 1.$

Problems

1. Show that the following standard triples of lines are concurrent: the medians (easy); the angle bisectors (medium); the altitudes (hard).

2. A circle inscribed in $\triangle ABC$ (the incircle) is tangent to BC at X, to AC at Y, to AB at Z. Show that AX, BY, and CZ are concurrent.

3. Three squares are drawn on the sides of $\triangle ABC$ (i.e. the square on AB has AB as one of its sides and lies outside $\triangle ABC$). Show that the lines drawn from the vertices A, B, C to the centers of the opposite squares are concurrent.

4. Prove that for any points X, Y, Z on BC, AC, AB,

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{\sin\angle ACZ}{\sin\angle ZCB} \cdot \frac{\sin\angle BAX}{\sin\angle XAC} \cdot \frac{\sin\angle CBY}{\sin\angle YBA}.$$

Solutions

1.
   1. For the medians, $\frac{AZ}{ZB} = \frac{BX}{XC} = \frac{CY}{YA} = 1$, so their product is 1.
   2. For the angle bisectors, use the angle bisector theorem:
   $$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{AC}{BC} \cdot \frac{AB}{AC} \cdot \frac{BC}{AB} = 1.$$
   3. For the altitudes, $\triangle ABX$ and $\triangle CBZ$ are similar, because $\angle ABX = \angle CBZ = \angle ABC$ and $\angle AXB = \angle CZB = 90°$. Therefore $\frac{BZ}{BX} = \frac{BC}{AB}$, which lets us simplify the Ceva's theorem product in the same way as above, after rearranging.
2. $AY = AZ$ because these are the two tangent lines from A to the incircle. Similarly, $BX = BZ$ and $CX = CY$, and the result follows.

Solutions

3. Let A', B', C' be the centers of the squares opposite A, B, C. Then $\triangle ABA'$ and $\triangle CBC'$ have the same area: $\angle ABA' = \angle CBC' = \angle ABC + 45°$, and $BA' : BC' = CB : AB$.

   In the proof of Ceva's theorem we had $\frac{BX}{XC} = \frac{[ABX]}{[ACX]}$, and by the same argument we have $\frac{BX}{XC} = \frac{[ABA']}{[ACA']}$. Therefore

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{[CAC']}{[CBC']} \cdot \frac{[ABA']}{[ACA']} \cdot \frac{[BCB']}{[BAB']} = 1$$

   as the three pairs of areas which we proved to be equal cancel.

4. Using the $\frac{1}{2}ab\sin\theta$ formula for the area of a triangle, we have

$$\frac{BX}{XC} = \frac{[AXB]}{[AXC]} \frac{\frac{1}{2}AC \cdot AX \cdot \sin\angle XAC}{\frac{1}{2}AB \cdot AX \cdot \sin\angle BAX} = \frac{AC}{AB} \cdot \frac{\sin\angle XAC}{\sin\angle BAX}.$$

   Doing the same for all three ratios yields the formula we want.