Collinearity

Varsity Practice 1/19/20

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1 Warm-Up Problems

1. Revisit similarity. Read the proofs of Menelaus's and Ceva's theorems.

2. In a quadrilateral $ABCD$, let $AD \cap BC = Q, AB \cap CD = P, AC \cap BD = R, QR \cap AB = K, PR \cap BC = L, AC \cap PQ = T$. Prove that K,L and T are collinear.
Hint: use Menelaus and Ceva ☺

3. Let ABC be an acute triangle and let M, N be distinct points on the segment BC such that $\angle BAM = \angle CAN$. If O_1, O_2 are the circumcenters of $\triangle ABC$ and $\triangle AMN$ respectively. Prove that A, O_1, O_2 are collinear.
Hint: don't use Menelaus and Ceva; angle chasing goes a long way: you might want to prove that $\angle O_2AB = \angle O_1AB$.

2 Problem Set

1. (Monge) Let C_1, C_2, C_3 be three circles and let their pairwise common tangents intersect at X, Y, Z. Prove that X, Y, Z are collinear.

2. (Simson) Given a point P in the plane of $\triangle ABC$. Suppose X, Y and Z are the feet of perpendiculars from P to BC, CA and AB respectively. Then X, Y and Z are collinear if and only if P lies on the circumcircle of $\triangle ABC$.

3. (Pappus) Let P_1, P_3, P_5 and P_2, P_4, P_6 be sets of collinear points (with P_1P_3 not parallel to P_2P_4). If $P_2P_3 \cap P_5P_6 = \{M_1\}, P_1P_6 \cap P_3P_4 = \{M_2\}, P_1P_2 \cap P_4P_5 = \{M_3\}$, show that M_1M_2, M_3 are collinear.

4. (Desargues) Let $A_1B_1C_1$ and $A_2B_2C_2$ be triangles. IF $\{P\} = B_1C_1 \cap B_2C_2, \{Q\} = A_1C_1 \cap A_2C_2, \{R\} = A_1B_1 \cap A_2B_2$, then P, Q, R are collinear if and only if A_1A_2, B_1B_2, C_1C_2 meet at the same point.

5. (JBMO 2016) A trapezoid $ABCD$ ($AB||CD$,$AB > CD$) is circumscribed.The incircle of the triangle ABC touches the lines AB and AC at the points M and N,respectively.Prove that the incenter of the trapezoid $ABCD$ lies on the line MN.

6. (IZHO 2013) Given a trapezoid $ABCD$ ($AD \parallel BC$) with $\angle ABC > 90^\circ$. Point M is chosen on the lateral side AB. Let O_1 and O_2 be the circumcenters of the triangles MAD and MBC, respectively. The circumcircles of the triangles MO_1D and MO_2C meet again at the point N. Prove that the line O_1O_2 passes through the point N.

7. (RMM 2015) Let ABC be a triangle, and let D be the point where the incircle meets side BC. Let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the angle bisector of $\angle BAC$.