Cyclic Quadrilaterals

Varsity Practice 1/26/20

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1 Warm-Up Problems

1. Let $\square ABCD$ be a cyclic quadilateral such that $AB = 6$, $BC = 4$, $CD = 2$ and $AD = 3$. Let $AC = 4$ then compute length of BD.
2. Let $\triangle ABC$ be such that $AB = 4$. Let circle tangent to $\overline{AC}$ at C and passing through B intersect $\overline{AB}$ at a point P such that $PB = 5$. Compute AC.
3. (PUMaC 2016 A5) Let D, E, F respectively be the feel of the altitudes from A,B and C of acute triangle $\triangle ABC$ such that $AF = 28$, $FB = 35$ and $BD = 45$. Let P be the point on segment BE such that $AP = 42$. Find the length of CP.
4. (IMO Shortlist 2017 G7) A convex quadrilateral $\square ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD, CDA respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90°$.

2 Problem Set

1. (AIME 2 2015) The circumcircle of acute $\triangle ABC$ has center O. The line passing though point O perpendicular to $\overline{OB}$ intersects lines AB and line BC at P and Q, respectively. Also $AB = 5, BC = 4, BQ = 4.5$ and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
2. (AIME 2, 2010) In triangle $\triangle ABC$, $AC = 13, BC = 14$ and $AB = 15$. Points M and D lie on AC wtih $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on AB with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets BC at Q. The ratio BQ/CQ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m - n$.
3. (PUMaC 2016 A8) Let $\triangle ABC$ have side lengths $AB = 4$, $BC = 6$, $CA = 5$. Let M be the midpoint of BC and let P be the point on the circumcentre of $\triangle ABC$ such that $\angle MPA = 90°$. Let D be the foot of the altitude from B to AC, and let E be the foot of the altitude from C to AB. Let PD and PE intersect line BC at X and Y respectively. Compute the square of the area of $\triangle AXY$.
4. (IMO shortlist 2016 G4) Let $\triangle ABC$ be a triangle with $AB = AC \neq BC$ and let I be its incenter. The line BI meets AC at D, and the line through D perpendicular to AC meet AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.