Cevians and Weighted Averages

Varsity Practice 2/16/20

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Warm-Up Problems

1. Look up any of the following terms you don't know: cevian, median, altitude, angle bisector, centroid, incenter.

2. Prove that the medians of a triangle divide the triangle into six regions of equal area.

3. (Mathcounts) In right triangle ABC, $AC = 4$ and $CB = 7$. Point E is the midpoint of AC. Point F lies on CB such that $CF : FB = 3 : 4$. Point D is the intersection of AF and EB. Compute the area of triangle ABD.

4. (AIME) Let P be an interior point of $\triangle ABC$ and extend lines from the vertices through P to the opposite sides. Given that $AP = a$, $BP = b$, $CP = c$, $PF = PE = PD = 3$, find the product abc if $a + b + c = 43$.

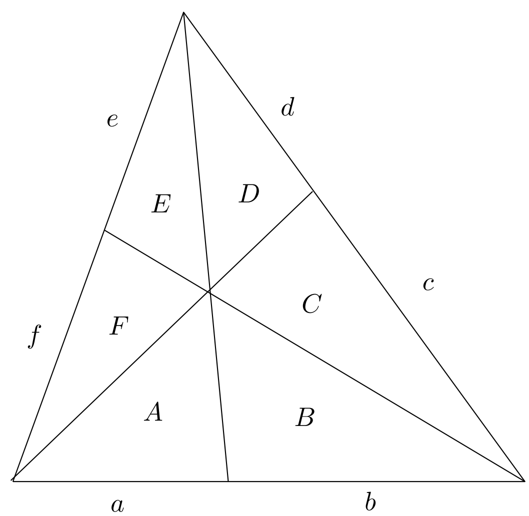
Set 1

1. In $\triangle ABC$, D is the midpoint of BC and E is the trisection point of AC nearer to A (i.e., $AE : EC = 1 : 2$). Let G be the intersection of BE and AD. Find $AG : GD$ and $BG : GE$.

2. (Mathcounts) In rectangle $ABCD$, point E lies on BC so that $\frac{BE}{EC} = 2$ and point F lies on DC so that $\frac{CF}{FD} = 2$. Line segments AE and AC intersect BF at points X and Y, respectively. Given that $FY : YX : XB = a : b : c$, where a, b, and c are relatively prime positive integers, compute the ordered triple (a, b, c).

3. The sides of $\triangle ABC$ are $AB = 13$, $BC = 15$, and $AC = 14$. Let BD be an altitude of the triangle. The angle bisector of $\angle C$ intersects side AB at F and altitude BD at E. Find $CE : EF$.

4. In the diagram below a, b, c, d, e, f are the lengths of the corresponding line segments and A, B, C, D, E, F are the areas of the corresponding triangles.



   (a) Prove that $ACE = BDF$.
   (b) Use part (a) to prove Ceva's Theorem, which states that $\frac{ace}{bdf} = 1$.
   (c) Use Ceva's Theorem to prove that the method of mass points works, i.e. that three cevians are concurrent at the center of mass if and only if the feet of the cevians are the centers of mass of the respective sides of the triangle.

5. Prove that the altitudes of an acute triangle are concurrent (i.e. they intersect in a single point). *This is also true of right and obtuse triangles.*

6. (AIME) Triangle ABC has side lengths $AB = 21$, $AC = 22$, and $BC = 20$. Points D and E are on AB and AC, respectively, such that DE is parallel to BC and contains the center of the inscribed circle of ABC. Compute DE.

Set 2

1. Triangle ABC has cevians AD, BE, and CF which are concurrent at point P. Prove that
$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PE}{CE} = 1.$$

2. (AIME) In parallelogram ABCD, point M is on AB so that $\frac{AM}{AB} = \frac{17}{1000}$ and and point N is on AD so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of AC and MN. Find $\frac{AC}{AP}$.

3. Let $ABCD$ be a quadrilateral with an inscribed circle. Let M on AB, N on BC, P on CD, and Q on DA be the points of tangency of the quadrilateral with the circle. Suppose that $AM = a$, $BN = b$, $CP = c$, and $DQ = d$. Let Z be the intersection of MP and NQ. Find the ratio $MZ : ZP$.

4. (Purple Comet) Triangle ABC has sides $AB = 39$, $BC = 57$, and $CA = 70$. Median AD is divided into three congruent segments by points E and F. Lines BE and BF intersect side AC at points G and H, respectively. Find GH.

5. Triangle ABC has area 143. Let A_1, B_1, C_1 be points on the sides of triangle ABC such that
$$\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{4}.$$
Find the area of the triangle PQR bounded by the lines AA_1, BB_1, and CC_1.

6. (AIME) In $\triangle ABC$, cevians AD, BE, and CF intersect at point P. The areas of $\triangle PAF$, $\triangle PFB$, $\triangle PBD$, and $\triangle PCE$ are 40, 30, 35, and 84, respectively. Find the area of $\triangle ABC$.

7. (AIME) Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.