Sequences

JV Practice 1/24/21

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Definitions

Monotone

Arithmetic Sequence

Geometric Sequence

Convergence

Warm-up Problems

1. Prove the sum of the first n terms of an arithmetic sequence is given by $S_n = a_1 + a_2 + \cdots + a_n = \frac{n}{2}(a_1 + a_n)$
2. Prove the sum of the first n terms of a geometric sequence is given by $S_n = a_1 + r * a_1 + r^2 * a_1 + \cdots + r^{n-1} * a_1 = a_1 \cdot \frac{r^n - 1}{r - 1}$
3. Prove the sum of an infinite geometric sequence is given by $S_n = a_1 + r * a_1 + r^2 * a_1 + \cdots = \frac{a_1}{1-r}$

Problems

1. (2006 Fermat #13) The numbers $4x, 2x-3, 4x-3$ are three consecutive terms in an arithmetic sequence. What is the value of x?
2. (2019 Fermat #11) The function f has the properties that $f(1) = 6$ and $f(2x+1) = 3f(x)$ for every integer x, what is the value of $f(63)$?
3. (2003 AMC 12A #1) What is the difference between the sum of the first 2003 even counting numbers and the sum of the first 2003 odd counting numbers?
4. (2005 Cayley #16) The non-negative difference between two numbers a, b is $a - b$ or $b - a$, whichever is greater than or equal to 0. For example, the non-negative difference between 24 and 64 is 40. In the sequence 88, 24, 64, 40, 24, , each number after the second is obtained by finding the non-negative difference between the previous 2 numbers. What is the sum of the first 100 numbers in this sequence?
5. (2012 Cayley #18) Six consecutive integers are written on a blackboard. When one of them is erased the sum of the remaining five integers is 2012. What is the sum of the digits of the integer that was erased?

6. (1975 AHSME #16) If the first term of an infinite geometric series is a positive integer, the common ratio is the reciprocal of a positive integer, and the sum of the series is 3, then what is the sum of the first two terms of the series?

7. (2004 AMC 10B #10) A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

8. (2006 AMC 12B #14) The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

9. (2006 AMC 12A #12) A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?

10. (2003 AIME I #2) One hundred concentric circles with radii $1, 2, 3, \ldots, 100$ are drawn in a plane. The interior of the circle of radius 1 is colored red, and each region bounded by consecutive circles is colored either red or green, with no two adjacent regions the same color. The ratio of the total area of the green regions to the area of the circle of radius 100 can be expressed as m/n, where m and n are relatively prime positive integers. Find $m + n$.

11. (2007 AIME II #12) The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that

    $\sum_{n=0}^{7} \log_3(x_n) = 308$ and $56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57$,

    find $\log_3(x_{14})$.

12. (Difficult Problem, generatingfunctionology) Given a recursive sequence $a_{n+1} = 2a_n + 1, n \geq 1, a_0 = 1$, find the explicit sequence of the form $a_n = 3n + 1$

13. (Very Difficult Problem, generatingfunctionology) Given a recursive sequence $a_{n+1} = 2a_n + n, n \geq 1, a_0 = 1$, find the explicit sequence of the form $a_n = 3n + 1$