Sequences

JV Practice 1/31/21

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Definitions

1. The sum of the first n terms of an arithmetic sequence is given by $S_n = a_1 + a_2 + \cdots + a_n = \frac{n}{2}(a_1 + a_n)$
2. The sum of the first n terms of a geometric sequence is given by $S_n = a_1 + r * a_1 + r^2 * a_1 + \cdots + r^{n-1} * a_1 = a_1 \cdot \frac{r^n - 1}{r-1}$
3. The sum of an infinite geometric sequence is given by $S_n = a_1 + r * a_1 + r^2 * a_1 + \cdots = \frac{a_1}{1-r}$

Warm-up Problems

1. (2016 Cayley #19) A total of points are equally spaced around a circle and are labelled with the integers 1 to n, in order. Two points are called diametrically opposite if the line segment joining them is a diameter of the circle. If the points labelled 7 and 35 are diametrically opposite, then what does n equals?
2. (2016 Cayley #25) A new language uses only the letters A, B, C, D, and E. The letters A and E are called vowels, while the letters B, C and D are called consonants. A sequence of letters is called a word if it does not include the same letter twice in a row, and it does not include two vowels in a row. How many words are there in this language that are 10 letters long and that begin with a vowel?

Problems

1. (2005 Fermat #6) What is the sum of the first 2005 terms of the sequence 1, 2, 3, 4, 1, 2, 3, 4 ?
2. (2005 Fermat #8) Seven children, each with the same birthday, were born in seven consecutive years. The sum of the ages of the youngest three children is 42. What is the sum of the ages of the oldest three?
3. (2005 Fermat #24) The arithmetic sequence $a, a + d, a + 2d, a + 3d \ldots a + (n - 1)d$ has the following properties:

   When the first, third, and fifth, and so on terms are added, up to and including the last term, the sum is 320.

   When the first, fourth, seventh, and so on, terms are added, up to and including the last term, the sum is 224.

   What is the sum of the whole sequence?

4. (2006 Fermat #17) In a seven term sequence,$5, p, q, 13, 4, 40, x$, each term after the third term is the sum of the preceding three terms. What is the value of x?

5. (2006 Fermat #18) If the first four terms of an arithmetic sequence are $a, 2a, b, a-6-b$ and for some numbers a and b, then what is the value of the 100th term?

6. (2006 AMC 12A #8) How many sets of two or more consecutive positive integers have a sum of 15?

7. (2015 Fermat #22) Three distinct integers a, b, c satisfy the following three conditions:

   $abc = 17,955$

   a, b, c forms an arithmetic sequence in that order

   $(3a + b), (3b + c), (3c + a)$ form a geometric sequence in that order. What is the value of $a + b + c$?

8. (2006 AMC 10A #19) How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

9. (2012 AMC 10A #22) The sum of the first m positive odd integers is 212 more than the sum of the first n positive even integers. What is the sum of all possible values of n?

10. (2014 AMC 10A #24) A sequence of natural numbers is constructed by listing the first 4, then skipping one, listing the next 5, skipping 2, listing 6, skipping 3, and, on the nth iteration, listing $n + 3$ and skipping n. The sequence begins $1, 2, 3, 4, 6, 7, 8, 9, 10, 13$. What is the 500,000th number in the sequence?

11. (2012 AMC 12A #17) Let $a + ar_1 + ar_1^2 + ar_1^3 + \cdots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $r_1 + r_2$?

12. (2012 AIME I #2) The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

13. (2005 AIME II #3) An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m + n$.

14. (2011 AIME II #5) The sum of the first 2011 terms of a geometric sequence is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.

15. (1985 AIME #5)A sequence of integers $a_1, a_2, a_3, \ldots$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of the first 2001 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492?

16. (1985 AIME #13) The numbers in the sequence 101, 104, 109, 116,... are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.