Number Theory and Modular Arithmetic

JV Practice 2/7/21

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1 Background

We consider a and b congruent modulo n, written as $a \equiv b \mod n$ when b is the reminder of a divided by n. In other words, n divides $a - b$. Modulo has the following basic properties given $a \equiv b \mod n$ and $c \equiv d (\mod n)$:

- $a + c \equiv b + d (\mod n)$
- $a - c \equiv b - d (\mod n)$
- $a \cdot c \equiv b \cdot d (\mod n)$

2 Warmup

1. Do the following computations in modular arithmetic:
   (a) $4 - 20 \mod 5$
   (b) $5 \cdot 11 \mod 12$
   (c) $2^5 \mod 7$
   (d) $11^{378} \mod 10$
2. For any non-negative integer n, the factorial $n! = n \cdot (n-1) \ldots 3 \cdot 2 \cdot 1$. What are the following quantities? Is there a similarity?
   (a) $6! \mod 7$
   (b) $4! \mod 5$
   (c) $12! \mod 13$
3. A number is divisible by 4 if and only if its last two digits are divisible by 4. Why?

3 Problems

1. Find the perfect squares in modulo 9
2. Find x such that $2x \equiv 23 (\mod 39)$
3. Given that $5x \equiv 6 (\mod 8)$ find x
4. Is there an x such that $6x \equiv 22 (\mod 39)$?
5. Show that $7 | (4^{2^n} + 2^{2^n} + 1)$ for any natural number n

6. Find the remainder when

$$\sum_{n=0}^{100} 10^n$$

is divided by 9

7. What is the largest positive integer n such that $n^3 + 100$ is divisible by $n + 10$

4 Challenge Problems

1. Show that if p is a prime, then $a^{p-1} \equiv 1 \mod p$

2. Show that $(p-1)! \equiv -1 \mod p$ whenever p is prime