Counting

JV Practice 3/14/21

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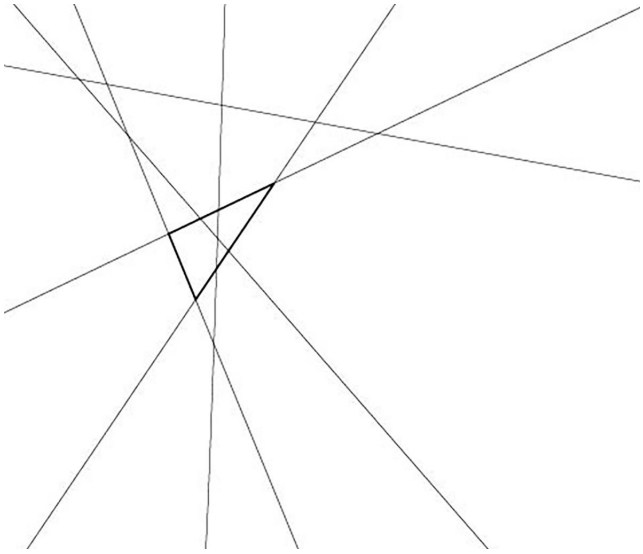
1 Warmups

1. How many triangles are there in this picture below?



2. How many rectangles are there on a 9×9 chess board?
3. There are 25 questions on the AMC exam and five possible answers to each question.
   (a) In how many ways can the exam be answered?
   (b) In how many ways can the exam be answered with exactly 18 answers correct?
   (c) In how many ways can the exam be answered with at least 3 answers correct?
4. Compute the coefficient of xy^3z^5 in $(x + y + z)^9$.

2 General tips for counting

- Draw a picture of the things or way you are counting, so you have a clear idea of what you're working with
- Figure out whether order matters
- Divide out by things you can rearrange, multiply in by the number of choices you make
- Decide whether counting all the things you want is easier, or if counting all the things you *don't* want is easier

3 Problems

1. In how many ways can 12 people form four groups of three if
   (a) the groups have names?
   (b) the groups are unnamed?
2. (AMC 10B 2010) How many palindromes are there between $1,000$ and $10,000$?
3. (AMC 10B 2010) How many palindromes between $1,000$ and $10,000$ are divisible by 7?
4. Rebecca, C.J. Elizabeth, Alex and Da Qi are coaching ARML practices for the day. Everyone goes to at least one of JV and Varsity, and exactly two of them will go to both practices. How many ways are there to decide who goes to each practice?
5. (AMC 10A 2013) Rabbits Peter and Pauline have three offspring - Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?
6. A pair (A, B) of sets is *groovy* if A and B are subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A \subseteq B$ (A is a subset of B). How many groovy pairs of sets are there?
7. If you wrote the integers from 1 to 2021, how many times would you write the digit 2?
8. (AMC 10A 2006) How many four-digit positive integers have at least one digit that is a 2 or a 3?
9. (AMC 10B 2007) A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?
10. (AMC 10A 2008) Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?
11. (AMC 12B 2008) A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

4 Somewhat Harder Problems

1. (AIME I 2002) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find (m, n).

2. (AIME II 2005) A game uses a deck of n different cards, where n is an integer and $n \geq 6$. The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. Find n.

3. (AIME I 2010) Define an ordered triple (A, B, C) of sets to be minimally intersecting if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1, 2\}, \{2, 3\}, \{1, 3, 4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Find N.

4. (AIME 1999) Ten points in the plane are given, with no three collinear. Four distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. The probability that some three of the segments form a triangle whose vertices are among the ten given points is m/n, where m and n are relatively prime positive integers. Find (m, n).