Alex Rudenko, 4rudenko@gmail.com

Counting in two ways

1 Warm-Up

1. Prove that
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

2. In each cell of a 5×5 square grid is written either $+1$ or -1. The product of the values in each row and each column is computed. Is it possible that the sum of these ten values is zero?

3. In a certain committee, each member belongs to exactly three subcommittees, and each subcommittee has exactly three members. Prove that the number of members equals to the number of subcommittees.

4. Let $a_1, a_2, \ldots, a_n$ be arbitrary positive integers. Let b_k be number of a_i's that are at least k. Prove that
$$a_1 + \ldots + a_n = b_1 + b_2 + \ldots$$

2 Identities

1. $$\binom{n}{k} = \binom{n}{n-k}$$

2. $$k\binom{n}{k} = n\binom{n-1}{k-1}$$

3. $$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

4. $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

5. $$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

6. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

7. $$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

8. $$\binom{n}{k} + \binom{n-1}{k} + \ldots = \binom{n+1}{k+1}$$

9.
$$\sum_{k=1}^{n} \binom{n}{k} \binom{m}{t-k} = \binom{n+m}{t}$$

10.
$$\sum_{k=1}^{n} \binom{n}{k} \binom{k}{m} = \binom{n}{m} 2^{n-m}$$

3 Problems

1. There are n points in the plane such that no three of them are collinear. Prove that the number of triangles, whose vertices are chosen from these n points and whose area is 1, is not greater than $\frac{2}{3}(n^2 - n)$.

2. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

3. In a school there are 2007 girls and 2007 boys. Each student joins no more than 100 clubs in the school. It is known that any two students of opposite genders have joined at least one common club. Show that there is a club with at least 11 boys and 11 girls.

4. A school has n students, and each student can take any number of classes. Every class has at least two students in it. We know that if two different classes have at least two common students, then the number of students in these two classes is different. Prove that the number of classes is not greater that $(n-1)^2$.