Complex Numbers

JV Practice 9/13/20

Zoe Wellner

If one is measuring flour for a cookie recipe, it would make sense to consider two and a half cups of flour or other fraction measurements. If NASA is planning the creation of lunar shuttles, it only makes sense to consider a whole shuttle, and not any fractional shuttles. In this way, we can see that different situations might require us to deal with different number systems. Up until this point, it is likely that the number system you have considered with the most frequency is the real numbers denoted $\mathbb{R}$, but this number system has some limitations. Our goal is to get more familiar with the "complex numbers," denoted $\mathbb{C}$ which is a lot like the reals but with a new number i.

Let $i = \sqrt{-1}$, and therefore $i^2 = -1$. This will allow us to represent quantities that we didn't in real numbers, like $\sqrt{-16} = 4i$, as valid complex numbers. We will consider $z = a + bi$ as the *standard form* of a complex number. We also will define the *complex conjugate* of the number $z = a + bi$ to be $\overline{z} = a - bi$.

1 Warmup

1. What do all of our usual operations (addition, multiplication, ect ...) look like in complex numbers?
2. What does a complex number look like?
3. Write each of the following expressions in standard form ($a+bi$ for some a and b real numbers):
   - $(-4 + 7i) + (5 - 10i)$
   - $(1 - 5i)(-9 + 2i)$
   - $\frac{3-i}{2+7i}$
4. Find all the roots (x satisfying the equation) of $2x^3 + 2x^2 + x - 5 = 0$.
5. Find c if a, b, and c are positive integers which satisfy $c = (a + bi)^3 - 107i$

2 Problems

1. Compute $|1 + 2i|^2$ and $(1 + 2i)^2$. Do the same for $|2 + 3i|^2$, $(2 + 3i)^2$. Do you notice anything special about the numbers you find?
2. If $\frac{(x+yi)}{i} = (7 + 9i)$, where x and y are real, what is the value of $(x + yi)(x - yi)$?
3. Determine all complex number z that satisfy the equation $z + 3z' = 5 - 6i$, where z' is the complex conjugate of z.
4. Find all complex numbers z such that $(4+2i)z+(8-2i)z' = -2+10i$, where z' is the complex conjugate of z.

5. Given that the complex number $z = -2+7i$ is a root to the equation: $z^3+6z^2+61z+106 = 0$, find the real root to the equation.

6. Prove that $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$ for all θ.

7. Find the number of ordered pairs of real numbers (a, b) such that $(a+bi)^{2002} = a - bi$.

8. Write the complex number $1-i$ in polar form. Then use DeMoivre's Theorem to write $(1-i)^{10}$ in the complex form $a+bi$, where a and b are real numbers and do not involve the use of a trigonometric function.

9. Find all of the solutions to the equation $x^3 - 1 = 0$.

10. (AMC 2017) There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?

11. (AIME 2009) There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.