Modular Arithmetic and Divisibility

Number Theory

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1 Solutions

Note: in these solutions, primes are assumed to be positive!

1. *Using modular arithmetic, show that 3 divides n if and only if 3 divides the sum of the digits of n. Do the same for 9. Can you find something similar for 11?*

   We only provide the solution for the case of 11. (The case for 3 was done in class and the case for 9 is identical.) We claim that 11 divides n if and only if 11 divides the alternating sum of the digits. We can represent $n = d_k 10^k + d_{k-1}10^{k-1} + \cdots + d_1 10^1 + d_0$. Now assume that 11 divides n or $0 \equiv n \pmod{11}$. Note that $10 \equiv -1 \pmod{11}$ so:

   $$n = d_k 10^k + d_{k-1}10^{k-1} + \cdots + d_1 10^1 + d_0 \equiv d_k(-1)^k + d_{k-1}(-1)^{k-1} + \cdots + d_1(-1)^1 + d_0 \pmod{11}$$

   Thus, 11 divides the alternating sum of digits as well. For the converse, assume that 11 divides $d_k(-1)^k + d_{k-1}(-1)^{k-1} + \cdots + d_1(-1)^1 + d_0$, the alternating sum of digits. Then:

   $$d_k(-1)^k + d_{k-1}(-1)^{k-1} + \cdots + d_1(-1)^1 + d_0 \equiv d_k 10^k + d_{k-1}10^{k-1} + \cdots + d_1 10^1 + d_0 = n \pmod{11}$$

   Thus, 11 divides n as well. Note that this actually proves something stronger–n is actually always congruent to its alternating sum of digits (mod 11) regardless of if $n \equiv 0 \pmod{11}$.

2. *Find* $\gcd(221, 299)$ *and* $\gcd(2520, 399)$.

   We calculate $\gcd(221, 299)$.

   $$\begin{aligned} \gcd(221,299) &= \gcd(221, 299-221) && 299 = 221 + 78 \\ \gcd(221,78) &= \gcd(78, 221 - 2\cdot 78) && 221 = 2\times 78 + 65 \\ \gcd(78,65) &= \gcd(65, 78-65) && 78 = 65 + 13 \\ \gcd(65,13) &= \gcd(13, 65 - 5\cdot 13) && 65 = 5 \times 13 + 0 \\ &= \gcd(13,0) \end{aligned}$$

   So $\gcd(221, 299) = 13$.

   We calculate $\gcd(2520, 399)$.

   $$\begin{aligned} \gcd(2520,399) &= \gcd(399, 2520 - 6\cdot 399) && 2520 = 6 \times 399 + 126 \\ \gcd(399,126) &= \gcd(399, 399 - 3\cdot 126) && 399 = 3\times 126 + 21 \\ \gcd(126,21) &= \gcd(21, 126 - 6\cdot 21) && 126 = 6 \times 21 + 0 && = \gcd(21,0) \end{aligned}$$

   So $\gcd(2520, 399) = 21$.

3. $333 + 999 \equiv 3 + 4 \equiv 7 \equiv 2 \pmod 5$.

   $3333 \times 7777 = 3 \times 2 \equiv 6 \equiv 1 \pmod 5$.

4. *How many steps does it take the Euclidean Algorithm to reach* $(1, 0)$ *when the input is* $(n + 1, n)$*?*

   We trace through the steps of the Euclidean Algorithm for $\gcd(n + 1, n)$:

$$\begin{aligned} \gcd(n+1, n) &= \gcd(n, n + 1 - n) & \qquad n + 1 &= (n) + 1 \\ \gcd(n, 1) &= \gcd(1, n - 1 \cdot n) & n &= 1 \times n + 0 \\ &= \gcd(1, 0) = 1 \end{aligned}$$

   So we see that this takes 2 steps.

5. *Let* n *be a positive integer. Construct a set of* n *consecutive positive integers that are not prime.*

   Let n be a positive integer. Then take $(n+1)! + 2, (n+1)! + 3, \ldots, (n+1)! + (n+1)$, which are n consecutive positive integers. Note that all of these integers are composite (not prime): for each $i = 2, \ldots, n + 1$, i divides $(n + i)! + i$. Then, note that since $\frac{(n+i)!}{i} > 0$, we have that $\frac{(n+i)!+i}{i} = \frac{(n+i)!}{i} + 1 > 1$. As such, $(n + i)! + i$ is the product of 2 integers where neither is 1, making it composite.

6. *Find all positive integers* n *such that* $(n + 1)$ *divides* $(n^2 + 1)$.

   Note that if $\gcd(n + 1, n^2 + 1) = n + 1$, then $(n + 1)$ divides $(n^2 + 1)$. We (kind of) use the Euclidean Algorithm.

$$\begin{aligned} \gcd(n^2 + 1, n + 1) &= \gcd(n + 1, n^2 + 1 - (n - 1)(n + 1)) \qquad n^2 + 1 = (n - 1)(n + 1) + 2 \\ &= \gcd(n + 1, 2) \end{aligned}$$

   From, here, we see that $\gcd(n + 1, 2) = n + 1$ means that $n + 1 = 2$ or $n = 1$ is the only such n.

   Alternate. Substitute $m = n+1$ (and note $m \geq 2$). Then $n^2+1 = (m-1)^2+1 = m^2-2m+2$. Since we want m to divide $m^2 - 2m + 2$ and m divides $m^2 - 2m$, we need m to divide 2. As $m \geq 2$, we must have $m = 2$ or $n = 1$.

7. *Find all primes in the form* $n^3 - 1$.

   Note that $n^3-1 = (n-1)(n^2+n+1)$. Since $n^2+n+1 \geq 0$ for all integers n. (Convince yourself of this!), if $n \leq 1$, then $n^3-1 \leq 0$ and cannot be prime. If $n \geq 3$ then $n^3-1 = (n-1)(n^2+n+1)$ with neither factor being 1. The only remaining case is $n = 2$, in which $n^3 - 1 = 7$, which is prime.

8. *What is the largest positive integer* n *for which* $(n + 10)$ *divides* $n^3 + 100$*?*

   This uses the same idea as problem 6. Let $m = n + 10$ and note that $m \geq 11$. Then $n^3 + 100 = (m - 10)^3 + 100 = m^3 - 30m^2 + 300m - 1000 + 100 = m^3 - 30m^2 + 300m - 900$. Since m divides the first 3 terms, it remains for m to divide 900. The largest such m is then 900, making the largest $n = 890$.

9. *Show that $\underbrace{1\dots1}_{91\ ones}$ is composite.*

We claim that 1111111 divides $\underbrace{1\dots1}_{\text{91 ones}}$. Note that $\underbrace{1\dots1}_{\text{91 ones}} = 1111111 \times (10^0 + 10^7 + 10^14 + \cdots + 10^84)$, which gives that $\underbrace{1\dots1}_{\text{91 ones}}$ is composite.

10. *A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not 100 (such as 2012). The 200^{th} anniversary of the birth of novelist Charles Dickens was celebrated on February 7, 2012, a Tuesday. On what day of the week was Dickens born?*

We first count the number of leap years between 1812, inclusive and 2012, exclusive. Among the 200 years, 50 are divisible by 4 of which 1900 is not a leap year but 2000 is. Thus, there are 49 leap years. Then, note that there are 365 days in a non-leap year, which is $1 \pmod 7$ (so every year, a date is shifted by 1 as a day in the week). Letting Sunday be $0 \pmod 7$, then Tuesday is $2 \pmod 7$. If n is the day of the week when Dickens was born, note that $n + 200 \cdot 1 + 49 \equiv 2 \pmod 7 \Rightarrow n + 4 \equiv 2 \pmod 7 \Rightarrow n \equiv 5 \pmod 7$, so Dickens was born on a Friday.

11. *What is the largest prime factor of 7999488?*

Note that this is $8000000 - 512 = 512(15625 - 1) = 512(15624) = 2^9 \times 2^3 \times 3^2 \times 7 \times 31$, so 31 is the largest prime factor.

12. *An n-digit number is cute if its n digits are an arrangement of the set $\{1, 2, \dots, n\}$ and its first k digits form an integer that is divisible by k, for $k = 1, 2, \dots, n$. For example, 321 is a cute 3-digit integer because 1 divides 3, 2 divides 32 and 3 divides 321. How many cute 6-digit numbers are there?*

We begin to construct the number $abcdef$. Note that $e = 5$ since 5 must divide $abcde$. Note that b, d, f are some permutation of $2, 4, 6$ since $ab, abcd, abcdef$ are also divisible by 2. As such a, c are some permutation of $1, 3$. Note that then since 3 divides abc, as shown in problem 1, 3 must divide $a + b + c = 1 + 3 + b = 4 + b$. As such, $b = 2$ is necessary. Then note that since 4 must divide $abcd$, by divisibility rules, 4 must divide cd.

- If $d = 4$, then c must be 2, but c must be odd, so this is impossible.
- If $d = 6$, then c can be either 1 or 3, and then taking a to be the remaining odd number and $f = 4$ works.

Thus, the only cute numbers are 123654 and 321654, and so there are 2 cute 6-digit numbers.

13. *An old receipt has faded. It reads 88 chickens at the total of \$$x4.2y$, where x and y are unreadable digits. How much did each chicken cost?*

Note that we want $x42y$ to be a 4-digit number divisible by 88. Since $x42y$ is divisible by 8, we know that by divisibility rules, $42y$ is divisible by 8, so we must have that $y = 4$. By divisibility rules for 11, note that $-x + 4 - 2 + y \equiv 0 \pmod{11} \Rightarrow -x + 2 + 4 \equiv 0 \pmod{11} \Rightarrow x \equiv 6 \pmod{11}$. As such, we must have $x = 6$ so the total cost was \$64.24 and the cost of one chicken was that \$0.74.

14. *Find the smallest positive integer such that $\frac{n}{2}$ is a square and $\frac{n}{3}$ is a cube.*

Note that clearly $n = 2^a 3^b$ for some natural numbers a, b is necessary for the smallest such n. By the conditions, $\sqrt{2^{a-1}3^b}$ must be an integer, so 2 must divide $a-1$ and b, and $\sqrt[3]{2^a 3^{b-1}}$ must be an integer, so 3 must divide a and $b-1$. Then $a = 3, b = 4$ are the smallest possible values for a, b (check that smaller values fail). Thus, 648 is the smallest positive integer.

15. and 16. were challenge problems. :)